



Worked solutions to the exercise sheet

Similar triangles: Enlargements and reductions

Difficulty: ★☆☆ very easy ★★☆☆ easy ★★★ medium ★★★★ hard

Track: 🏃 core 🎯 recommended 🏠 going further

The shaded boxes 📝 show how to write out solutions on a test paper.

To remember. Similar triangles. Side lengths proportional $\iff$ angles pairwise of equal measure (the symbol $\iff$ reads “if and only if”), and two pairs of angles suffice. Opposite angles of equal measure: corresponding sides.

The three steps. Match (using the angles) / set up (table of corresponding sides) / solve (k or cross-multiplication), then check (three equal ratios, large/small consistency).

Enlargement–reduction with scale factor k . Lengths and perimeters multiplied by k ; angle measures preserved; areas multiplied by k^2 ; volumes multiplied by k^3 . In reverse: the ratio of the areas is k^2 .

📝 **Fluency answers.** a) $180^\circ - 47^\circ - 86^\circ = 47^\circ$. b) $180^\circ - 90^\circ - 26^\circ = 64^\circ$. c) yes: third angles 80° and 40° , angles pairwise of equal measure. d) $3^2 = 9$. e) $2^3 = 8$. f) $k = 5$ (since $5^2 = 25$). g) $\frac{3}{4,5} = \frac{30}{45} = \frac{2}{3}$. h) $7 \times 0,5 = 3,5$ cm. i) $15 \times 1,2 = 18$ cm. j) $3 \times 100 = 300$ cm = 3 m.

📝 **Warm-up answers.** 1) $360 = 2^3 \times 3^2 \times 5$. 2) $\frac{54}{36} = \frac{18 \times 3}{18 \times 2} = \frac{3}{2}$. 3) $\frac{3}{5} \times \frac{10}{9} = \frac{30}{45} = \frac{2}{3}$. 4) $0,5^3 = 0,125$; $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. 5) $1,25 \times 10^4 = 12\,500$.

1. Recognising similar triangles

1.1. Similar or not? (using angles) 🏃

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Method. Associated models: **R1** (questions a) to f)); **R3** (question e)).

📝 a) In triangle ABC , the third angle measures: $\widehat{C} = 180^\circ - \widehat{A} - \widehat{B} = 180^\circ - 47^\circ - 86^\circ = 47^\circ$.

In triangle DEF , the third angle measures: $\widehat{F} = 180^\circ - \widehat{D} - \widehat{E} = 180^\circ - 47^\circ - 86^\circ = 86^\circ$.

The angles of ABC are 47° , 47° and 86° ; those of DEF are 47° , 47° and 86° .

The two triangles have angles pairwise of equal measure: by the angle characterisation, they are similar.

b) In triangle ABC , the third angle measures: $\widehat{C} = 180^\circ - \widehat{A} - \widehat{B} = 180^\circ - 35^\circ - 72^\circ = 73^\circ$.

In triangle DEF , the third angle measures: $\widehat{F} = 180^\circ - \widehat{D} - \widehat{E} = 180^\circ - 73^\circ - 35^\circ = 72^\circ$.

The angles of ABC are 35° , 72° and 73° ; those of DEF are 35° , 72° and 73° .

The two triangles have angles pairwise of equal measure: by the angle characterisation, they are similar.

📝 c) In triangle ABC , the third angle measures: $\widehat{C} = 180^\circ - \widehat{A} - \widehat{B} = 180^\circ - 50^\circ - 60^\circ = 70^\circ$.

In triangle DEF , the third angle measures: $\widehat{F} = 180^\circ - \widehat{D} - \widehat{E} = 180^\circ - 60^\circ - 65^\circ = 55^\circ$.

The angles of ABC are 50° , 60° and 70° ; those of DEF are 55° , 60° and 65° .

The two triangles do not have their angles pairwise of equal measure: by the angle characterisation, they are not similar.

d) ABC is right-angled at A , so its third angle measures $\widehat{C} = 180^\circ - 90^\circ - 32^\circ = 58^\circ$.

DEF is right-angled at D , so its third angle measures $\widehat{F} = 180^\circ - 90^\circ - 58^\circ = 32^\circ$.

The angles of ABC are 32° , 58° and 90° ; those of DEF are 32° , 58° and 90° .

The two triangles have angles pairwise of equal measure: by the angle characterisation, they are similar.

📝 e) In triangle ABC , the third angle measures: $\widehat{C} = 180^\circ - \widehat{A} - \widehat{B} = 180^\circ - 44^\circ - 68^\circ = 68^\circ$.

DEF is isosceles with apex D and its apex angle measures 92° : its two base angles, of equal measure, are each $\widehat{E} = \widehat{F} = (180^\circ - 92^\circ) \div 2 = 88^\circ \div 2 = 44^\circ$.

The angles of ABC are 44° , 68° and 68° ; those of DEF are 44° , 44° and 92° .

The two triangles do not have their angles pairwise of equal measure: by the angle characterisation, they are not similar.

f) ABC is equilateral: its three angles have the same measure and are each $\widehat{A} = \widehat{B} = \widehat{C} = 180^\circ \div 3 = 60^\circ$.

In triangle DEF , the third angle measures: $\widehat{F} = 180^\circ - \widehat{D} - \widehat{E} = 180^\circ - 60^\circ - 60^\circ = 60^\circ$.

The angles of ABC are 60° , 60° and 60° ; those of DEF are 60° , 60° and 60° .

The two triangles have angles pairwise of equal measure: by the angle characterisation, they are similar.

Warning! In a), the given angles, $\widehat{A} = 47^\circ$ and $\widehat{B} = 86^\circ$ in ABC , $\widehat{D} = 47^\circ$ and $\widehat{E} = 47^\circ$ in DEF , look different: it is the third angle that decides. Only compare the triangles after calculating it.

1.2. Similar or not? (using sides)

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Method. Associated model: **R2**.

 a) First pair. I order the sides from shortest to longest. In ABC : $AB = 3$; $BC = 4$; $AC = 5$. In DEF : $DE = 6$; $EF = 8$; $DF = 10$. I calculate the three ratios of sides of the same rank, from ABC to DEF :

$$\frac{DE}{AB} = \frac{6}{3} = 2 \quad \frac{EF}{BC} = \frac{8}{4} = 2 \quad \frac{DF}{AC} = \frac{10}{5} = 2$$

We have $\frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC} = 2$: the three ratios are equal, the side lengths are proportional, so the triangles are similar, with scale factor $k = 2$ from ABC to DEF (enlargement).

 Second pair. I order the sides from shortest to longest. In ABC : $AB = 3$; $BC = 4$; $AC = 5$. In DEF : $DE = 4$; $EF = 5$; $DF = 6$. I calculate the three ratios of sides of the same rank, from ABC to DEF :

$$\frac{DE}{AB} = \frac{4}{3} \quad \frac{EF}{BC} = \frac{5}{4} = 1,25 \quad \frac{DF}{AC} = \frac{6}{5} = 1,2$$

We have $\frac{EF}{BC} \neq \frac{DF}{AC}$. The ratios 1,25 and 1,2 are different: the side lengths are not proportional, so the triangles are not similar.

 Third pair. I order the sides from shortest to longest. In ABC : $AB = 4,5$; $BC = 6$; $AC = 7,5$. In DEF : $DE = 3$; $EF = 4$; $DF = 5$. I calculate the three ratios of sides of the same rank, from ABC to DEF :

$$\frac{DE}{AB} = \frac{3}{4,5} = \frac{30}{45} = \frac{2}{3} \quad \frac{EF}{BC} = \frac{4}{6} = \frac{2}{3} \quad \frac{DF}{AC} = \frac{5}{7,5} = \frac{50}{75} = \frac{2}{3}$$

We have $\frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC} = \frac{2}{3}$: the three ratios are equal, the side lengths are proportional, so the triangles are similar, with scale factor $k = \frac{2}{3}$ from ABC to DEF (reduction).

 b) I order the sides from shortest to longest. In ABC : $AB = 5$ cm; $BC = 6,5$ cm; $AC = 8$ cm. In EFG : $EF = 1$ cm; $FG = 1,2$ cm; $EG = 1,6$ cm. I calculate the three ratios of sides of the same rank, from ABC to EFG :

$$\frac{EF}{AB} = \frac{1}{5} = 0,2 \quad \frac{FG}{BC} = \frac{1,2}{6,5} = \frac{12}{65} \quad \frac{EG}{AC} = \frac{1,6}{8} = 0,2$$

We have $\frac{EF}{AB} = \frac{EG}{AC} = 0,2$, but $\frac{FG}{BC} = \frac{12}{65} \neq 0,2$ (since $0,2 \times 6,5 = 1,3 \neq 1,2$). Thus, $\frac{EF}{AB} = \frac{EG}{AC} \neq \frac{FG}{BC}$: the side lengths are not proportional, so EFG is not a reduction of ABC .

For triangle EFG to be a reduction with scale factor 0,2, we would need $\frac{FG}{BC} = 0,2$, that is $FG = 0,2 \times BC = 0,2 \times 6,5 = 1,3$ cm. I therefore replace $FG = 1,2$ cm with $FG = 1,3$ cm. I check the three ratios, from ABC to EFG :

$$\frac{EF}{AB} = \frac{1}{5} = 0,2 \quad \frac{FG}{BC} = \frac{1,3}{6,5} = 0,2 \quad \frac{EG}{AC} = \frac{1,6}{8} = 0,2$$

The three ratios are then equal to 0,2: EFG is a reduction of ABC , with scale factor $k = 0,2$.

To remember. Another modification works: replacing $BC = 6,5$ cm with $BC = 6$ cm, since $1,2 \div 0,2 = 6$. All three ratios are then also equal to 0,2.

1.2. Optional extension

Detailed solution: [addition and preservation of shape](#)

A1. Distinguishing differences and quotients

Each length increases by 1, but its multiplier depends on its initial value:

Initial sides	New sides	Quotient 1	Quotient 2	Quotient 3
(3; 4; 5)	(4; 5; 6)	$4/3$	$5/4$	$6/5$
(5; 5; 8)	(6; 6; 9)	$6/5$	$6/5$	$9/8$
(4; 4; 4)	(5; 5; 5)	$5/4$	$5/4$	$5/4$

A2. Checking the objects before comparing them

The lists are ordered and positive. We check $4 + 5 > 6$, $6 + 6 > 9$ and $5 + 5 > 5$: these are indeed three non-degenerate triangles.

For the first pair, $4/3 \neq 5/4$, since the cross-products $4 \times 4 = 16$ and $3 \times 5 = 15$ differ. For the second, $6/5 \neq 9/8$, because $6 \times 8 = 48 \neq 45 = 5 \times 9$. Two different quotients are enough to rule out proportionality; these two pairs are therefore not similar.

For the third pair, all three quotients equal $5/4$: the triangles are similar, with scale factor $5/4$ (enlargement). Two sides of equal length were not enough in the second row; here, all three have the same length.

A3. Moving from trials to a general question

The conjecture is: “adding the same positive length to all three sides preserves the shape exclusively for equilateral triangles”. The trials suggest this, but they cover neither all initial lengths nor all positive additions. We must reason with arbitrary numbers a, b, c, t satisfying the conditions of the problem.

B1. Positive addition preserves the existence of the triangle

Since a, b, c, t are positive, the three new sides are positive. Adding the same number preserves the order: $a + t \leq b + t \leq c + t$.

The initial triangle gives $a + b > c$. By adding $2t$, and then using $t > 0$, we obtain

$$(a + t) + (b + t) = a + b + 2t > c + 2t > c + t.$$

The sum of the two shorter new sides is strictly greater than the longest: a new non-degenerate triangle exists. This does not yet prove its similarity to the first.

B2. What similarity requires

The sides ordered in the same way must be proportional. Since a and b are non-zero,

$$\frac{a + t}{a} = \frac{b + t}{b} \implies b(a + t) = a(b + t) \implies ab + bt = ab + at.$$

Subtracting ab from both sides gives: $bt = at$. We can then divide by t , since $t > 0$, which gives $b = a$.

With sides a and c , the same approach gives $c(a + t) = a(c + t)$, then $ct = at$ and $c = a$. Thus $a = b = c$: any solution is an equilateral triangle. We have just proved a necessary condition.

B3. Checking that the condition is sufficient

Conversely, for an equilateral triangle of side $a > 0$, the three new sides are equal to $a + t$. They form an equilateral triangle and the three ratios are equal to

$$k = \frac{a + t}{a} = 1 + \frac{t}{a} > 1.$$

They are therefore indeed similar. The condition is sufficient as well: for any common increase $t > 0$, the new triangle is similar to the initial triangle if and only if the initial triangle is equilateral. This conclusion answers all permitted cases, not just the three trials.

C1. The excluded case $t = 0$

With $t = 0$, no side changes: every triangle is congruent, and therefore similar, to itself, with $k = 1$. All triangles work. In B2, we would obtain $b \times 0 = a \times 0$, which is true without requiring $a = b$. Dividing by zero is forbidden: that is the exact step that used the hypothesis $t > 0$.

C2. Positive lengths may fail to form a triangle

Subtracting 2 gives $(1; 2; 3)$. The lengths are positive, but $1 + 2 = 3$: the segments can only form a degenerate configuration. There is therefore no non-degenerate triangle to compare with the first.

In B1, the increase reinforced the triangle inequality: we added $2t$ to the sum of the shorter sides, compared with t to the longest side. Here, we subtract 4 from that sum and only 2 from the longest side. The strict inequality may disappear. Checking the geometric conditions comes before checking the quotients.

1.3. Justifying similarity: three configurations

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Method. Associated model: **D2**.

a) The angle at A is common to triangles AMN and ABC (with $M \in [AB]$ and $N \in [AC]$), so $\widehat{MAN} = \widehat{BAC}$.

The parallel lines (MN) and (BC) are intersected by the transversal (AB) at M and B respectively. The corresponding angles $\widehat{AMN}$ and $\widehat{ABC}$ therefore have the same measure: $\widehat{AMN} = \widehat{ABC}$.

Now, if two triangles have two pairs of angles of equal measure, then they are similar.

Therefore triangles AMN and ABC are similar, the corresponding vertices being $A \leftrightarrow A$, $M \leftrightarrow B$ and $N \leftrightarrow C$. ■

b) The angles $\widehat{AOB}$ and $\widehat{COD}$ are vertically opposite (with the lines (AC) and (BD) intersecting at O), so $\widehat{AOB} = \widehat{COD}$.

From the markings in the problem, we know that $\widehat{OAB} = \widehat{OCD}$.

Now, if two triangles have two pairs of angles of equal measure, then they are similar.

Therefore triangles OAB and OCD are similar, the corresponding vertices being $O \leftrightarrow O$, $A \leftrightarrow C$ and $B \leftrightarrow D$. ■

c) The angle at A is common to triangles ADE and ACB (with $D \in [AB]$ and $E \in [AC]$), so $\widehat{DAE} = \widehat{CAB}$.

From the markings in the problem, we know that $\widehat{ADE} = \widehat{ACB}$.

Now, if two triangles have two pairs of angles of equal measure, then they are similar.

Therefore triangles ADE and ACB are similar, the corresponding vertices being $A \leftrightarrow A$, $D \leftrightarrow C$ and $E \leftrightarrow B$. ■

Warning! In c), the matching is crossed: D , lying on $[AB]$, corresponds to C , and E , lying on $[AC]$, corresponds to B . It is the angles of equal measure that decide, never the position of the letters on the figure.

1.4. The butterfly configuration

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Method. Associated models: **D2** (question a)); **A1** (question a)); **L1** (questions b) and c)).

a) The angles $\widehat{DAE}$ and $\widehat{BAC}$ are vertically opposite (with the lines (BD) and (CE) intersecting at A), so $\widehat{DAE} = \widehat{BAC}$.

The parallel lines (DE) and (BC) are intersected by the transversal (BD) at D and B respectively. The alternate interior angles $\widehat{ADE}$ and $\widehat{ABC}$ therefore have the same measure: $\widehat{ADE} = \widehat{ABC}$.

Now, if two triangles have two pairs of angles of equal measure, then they are similar.

Therefore triangles ADE and ABC are similar, the corresponding vertices being $A \leftrightarrow A$, $D \leftrightarrow B$ and $E \leftrightarrow C$. ■

The corresponding sides $[AD]$ and $[AB]$ join the corresponding vertices $A \leftrightarrow A$ and $D \leftrightarrow B$.

The scale factor from ABC to ADE is $k = \frac{AD}{AB} = \frac{4,8}{3,6} = \frac{48}{36} = \frac{4}{3}$.

We have $k > 1$, so triangle ADE is an enlargement of triangle ABC , with scale factor $k = \frac{4}{3}$.

b) From question a), $A \leftrightarrow A$, $D \leftrightarrow B$ and $E \leftrightarrow C$: the corresponding sides are $[AD] \leftrightarrow [AB]$, $[AE] \leftrightarrow [AC]$ and $[DE] \leftrightarrow [BC]$.

I arrange the corresponding sides in a table:

sides of ABC	$AB = 3,6$	$AC = 2,4$	BC
sides of ADE	$AD = 4,8$	$AE = ?$	DE

From question a), the scale factor from ABC to ADE is $k = \frac{4}{3}$.

To go to triangle ADE , I multiply by k : $AE = \frac{4}{3} \times AC = \frac{4}{3} \times 2,4 = \frac{4 \times 2,4}{3} = \frac{9,6}{3} = 3,2 \text{ cm}$.

Check: $\frac{4,8}{3,6} = \frac{3,2}{2,4} = \frac{4}{3}$: the ratios are equal. The scale factor is strictly greater than 1, so each side of ADE must be strictly longer than its corresponding side in ABC : this is the case, $4,8 > 3,6$ and $3,2 > 2,4$. ✓

c) I complete the table from question b):

sides of ABC	$AB = 3,6$	$AC = 2,4$	$BC = ?$
sides of ADE	$AD = 4,8$	$AE = 3,2$	$DE = 5,2$

From question a), the scale factor from ABC to ADE is $k = \frac{4}{3}$.

To go back to triangle ABC , I divide by k ; dividing by $\frac{4}{3}$ means multiplying by its reciprocal $\frac{3}{4}$:

$$BC = DE \div \frac{4}{3} = 5,2 \times \frac{3}{4} = \frac{5,2 \times 3}{4} = \frac{15,6}{4} = 3,9 \text{ cm}$$

Check: $\frac{4,8}{3,6} = \frac{3,2}{2,4} = \frac{5,2}{3,9} = \frac{4}{3}$: the ratios are equal. The scale factor is strictly greater than 1, so each side of ADE must be strictly longer than its corresponding side in ABC : this is the case, $4,8 > 3,6$, $3,2 > 2,4$ and $5,2 > 3,9$. ✓

To remember. In the butterfly configuration with parallel wings, the two pairs of angles of equal measure always come from the same sources: the vertically opposite angles, and the alternate interior angles formed by the two parallel lines.

1.5. True or false?

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Method. Associated models: **T2** (questions a) to g)); **R3** (questions d) and e)).

a) TRUE.

In an equilateral triangle, all three angles have the same measure and are each equal to $180^\circ \div 3 = 60^\circ$. Two equilateral triangles therefore have their angles pairwise of equal measure, 60° , 60° and 60° : by the angle characterisation, they are always similar. ■

b) FALSE.

Counterexample: two isosceles triangles whose apex angles measure 120° and 90° . The base angles are each $(180^\circ - 120^\circ) \div 2 = 60^\circ \div 2 = 30^\circ$ in the first, and $(180^\circ - 90^\circ) \div 2 = 90^\circ \div 2 = 45^\circ$ in the second: their angles are 120° , 30° , 30° and 90° , 45° , 45° . They are both isosceles, but they do not have their angles pairwise of equal measure: they are not similar.

c) FALSE.

Counterexample: a right-angled triangle that has a 45° angle, and a right-angled triangle that has a 30° angle. Their third angles are $180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $180^\circ - 90^\circ - 30^\circ = 60^\circ$: their angles are 90° , 45° , 45° and 90° , 30° , 60° . They are both right-angled, but they do not have their angles pairwise of equal measure: they are not similar.

d) TRUE.

If the right angle is the apex angle, the two base angles are each $(180^\circ - 90^\circ) \div 2 = 90^\circ \div 2 = 45^\circ$: the angles are 90° , 45° and 45° .

If the right angle were a base angle, the other base angle would also measure 90° : these two angles would already add up to 180° , and nothing would be left for the apex angle. This case is impossible.

Only one case is possible: the angles are necessarily 90° , 45° and 45° . Any two such triangles have their angles pairwise of equal measure: by the angle characterisation, they are always similar. ■

✎ e) FALSE.

If the 80° angle is the apex angle, the two base angles are each $(180^\circ - 80^\circ) \div 2 = 50^\circ$: the angles are 80° , 50° and 50° .

If the 80° angle is a base angle, the other base angle also measures 80° , and the apex angle is $180^\circ - 80^\circ - 80^\circ = 20^\circ$: the angles are 80° , 80° and 20° .

Two cases are possible, with different angles. Counterexample: the two isosceles triangles obtained, with angles 80° , 50° , 50° and 80° , 80° , 20° . Each of them has an 80° angle, but they do not have their angles pairwise of equal measure: they are not similar.

✎ f) FALSE.

Counterexample: an equilateral triangle of side 2 cm and an equilateral triangle of side 5 cm. Their angles all measure 60° : by the angle characterisation, they are similar. But their sides do not have the same length: they are not congruent.

✎ g) TRUE.

Two congruent triangles have pairs of sides of equal length and pairs of angles of equal measure. By the angle criterion for similarity, they are always similar. ■

2. Calculating lengths

2.1. First calculation, two ways

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Method. Associated models: **A1** (question a)); **L1** (questions b) and c)).

✎ a) The corresponding sides $[RS]$ and $[UV]$ join the corresponding vertices $R \leftrightarrow U$ and $S \leftrightarrow V$.

The scale factor from RST to UVW is $k = \frac{UV}{RS} = \frac{6}{4} = 1,5$.

We have $k > 1$, so triangle UVW is an enlargement of triangle RST , with scale factor $k = 1,5$.

✎ b) From the problem statement, $R \leftrightarrow U$, $S \leftrightarrow V$ and $T \leftrightarrow W$: the corresponding sides are $[RS] \leftrightarrow [UV]$, $[RT] \leftrightarrow [UW]$ and $[ST] \leftrightarrow [VW]$.

I arrange the corresponding sides in a table:

sides of RST	$RS = 4$	$RT = 8$	$ST = 6$
sides of UVW	$UV = 6$	$UW = ?$	$VW = ?$

From question a), the scale factor from RST to UVW is $k = 1,5$.

To go to triangle UVW , I multiply by k : $VW = 1,5 \times ST = 1,5 \times 6 = 9$.

By cross-multiplication: in the table, $\frac{VW}{ST} = \frac{UV}{RS}$, that is $\frac{VW}{6} = \frac{6}{4}$. Cross-multiplication gives $4 \times VW = 6 \times 6$, so $VW = \frac{6 \times 6}{4} = \frac{36}{4} = 9$. I get the same result.

Here, the scale factor is faster: $k = 1,5$ is a simple decimal number, a single multiplication is enough, whereas cross-multiplication requires a multiplication followed by a division.

Check: $\frac{6}{4} = \frac{9}{6} = 1,5$: the ratios are equal. The scale factor is strictly greater than 1, so each side of UVW must be strictly longer than its corresponding side in RST : this is the case, $6 > 4$ and $9 > 6$. ✓

✎ c) I complete the table from question b):

sides of RST	$RS = 4$	$RT = 8$	$ST = 6$
sides of UVW	$UV = 6$	$UW = ?$	$VW = 9$

From question a), the scale factor from RST to UVW is $k = 1,5$.

To go to triangle UVW , I multiply by k : $UW = 1,5 \times RT = 1,5 \times 8 = 12$.

Check: $\frac{6}{4} = \frac{12}{8} = \frac{9}{6} = 1,5$: the ratios are equal. The scale factor is strictly greater than 1, so each side of UVW must be strictly longer than its corresponding side in RST : this is the case, $6 > 4$, $12 > 8$ and $9 > 6$. ✓

✎ d) The perimeter of triangle RST is $RS + RT + ST = 4 + 8 + 6 = 18$.

The perimeter of triangle UVW is $UV + UW + VW = 6 + 12 + 9 = 27$.

I notice that $1,5 \times 18 = 27$: the perimeter is multiplied by k . This was predictable: the perimeter is a sum of lengths, and each length is multiplied by $k = 1,5$, so their sum is too.

2.2. The Brevet (French national exam at the end of Year 10) problem

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Method. Associated models: **D1** (question a)); **D2** (question b)); **L1** (question c)).

✎ a) We know that the lines (AE) and (CD) are parallel, and that the line (AB) intersects them at A and C ; the angles $\widehat{CAE}$ and $\widehat{BCD}$ are corresponding angles.

Now, if two parallel lines are cut by a transversal, then the corresponding angles they form are equal.

Therefore $\widehat{BCD} = \widehat{CAE}$. ■

✎ b) We know that $\widehat{CAE} = \widehat{BCD}$, from question a).

Similarly, the parallel lines (CE) and (BD) are cut by the transversal (AB) at C and B respectively. The corresponding angles $\widehat{ACE}$ and $\widehat{CBD}$ therefore have the same measure: $\widehat{ACE} = \widehat{CBD}$.

Now, if two triangles have two pairs of equal angles, then they are similar.

Therefore triangles ACE and CBD are similar, the corresponding vertices being $A \leftrightarrow C$, $C \leftrightarrow B$ and $E \leftrightarrow D$. ■

c) From question b), $A \leftrightarrow C$, $C \leftrightarrow B$ and $E \leftrightarrow D$: the corresponding sides are $[AC] \leftrightarrow [CB]$, $[AE] \leftrightarrow [CD]$ and $[CE] \leftrightarrow [BD]$. The point C lies on the line segment $[AB]$, so $CB = AB - AC = 7,5 - 3 = 4,5$. I arrange the corresponding sides in a table:

sides of CBD	$CB = 4,5$	$CD = 9$	$BD = 5,4$
sides of ACE	$AC = 3$	$AE = ?$	$CE = ?$

The scale factor from CBD to ACE is $k = \frac{AC}{CB} = \frac{3}{4,5} = \frac{30}{45} = \frac{2}{3}$ (reduction).

To go to triangle ACE , I multiply by k :

$$AE = \frac{2}{3} \times CD = \frac{2}{3} \times 9 = \frac{2 \times 9}{3} = \frac{18}{3} = \boxed{6}$$

$$CE = \frac{2}{3} \times BD = \frac{2}{3} \times 5,4 = \frac{2 \times 5,4}{3} = \frac{10,8}{3} = \boxed{3,6}$$

The sides of triangle ACE therefore measure $AC = 3$, $AE = 6$ and $CE = 3,6$.

Check: $\frac{3}{4,5} = \frac{6}{9} = \frac{3,6}{5,4} = \frac{2}{3}$: the ratios are equal. The scale factor is strictly less than 1, so each side of ACE must be strictly shorter than its corresponding side in CBD : this is the case, $3 < 4,5$, $6 < 9$ and $3,6 < 5,4$. ✓

Warning! $[AB]$ is not a side of either of the two triangles: the quotient $\frac{AC}{AB} = \frac{3}{7,5} = 0,4$ is not the scale factor.

2.3. Corresponding sides: two configurations

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Method. Associated models: **D2** (questions a) and d)); **L1** (questions b), c), e) and f)).

a) From the problem statement, we know that $\widehat{A} = \widehat{D}$.

Similarly, we know that $\widehat{B} = \widehat{E}$.

Now, if two triangles have two pairs of equal angles, then they are similar.

Therefore triangles ABC and DEF are similar, the corresponding vertices being $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$. ■

b) From question a), $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$: the corresponding sides are $[AB] \leftrightarrow [DE]$, $[AC] \leftrightarrow [DF]$ and $[BC] \leftrightarrow [EF]$. I arrange the corresponding sides in a table:

sides of ABC	$AB = 6$	$AC = 9$	$BC = 7,5$
sides of DEF	$DE = 4$	$DF = ?$	$EF = ?$

The scale factor from ABC to DEF is $k = \frac{DE}{AB} = \frac{4}{6} = \frac{2}{3}$ (reduction).

c) To go to triangle DEF , I multiply by k :

$$EF = \frac{2}{3} \times BC = \frac{2}{3} \times 7,5 = \frac{2 \times 7,5}{3} = \frac{15}{3} = \boxed{5}$$

$$DF = \frac{2}{3} \times AC = \frac{2}{3} \times 9 = \frac{2 \times 9}{3} = \frac{18}{3} = \boxed{6}$$

Check: $\frac{4}{6} = \frac{6}{9} = \frac{5}{7,5} = \frac{2}{3}$: the ratios are equal. The scale factor is strictly less than 1, so each side of DEF must be strictly shorter than its corresponding side in ABC : this is the case, $4 < 6$, $6 < 9$ and $5 < 7,5$. ✓

d) The angle at A is shared by triangles ADE and ACB (with $D \in [AB]$ and $E \in [AC]$), so $\widehat{DAE} = \widehat{CAB}$.

From the problem statement, we likewise know that $\widehat{ADE} = \widehat{ACB}$.

Now, if two triangles have two pairs of equal angles, then they are similar.

Therefore triangles ADE and ACB are similar, the corresponding vertices being $A \leftrightarrow A$, $D \leftrightarrow C$ and $E \leftrightarrow B$. ■

e) From question d), $A \leftrightarrow A$, $D \leftrightarrow C$ and $E \leftrightarrow B$: the corresponding sides are $[AD] \leftrightarrow [AC]$, $[AE] \leftrightarrow [AB]$ and $[DE] \leftrightarrow [CB]$.

f) I arrange the corresponding sides in a table:

sides of ACB	$AC = 10$	$AB = 8$	$CB = 7,5$
sides of ADE	$AD = 4$	$AE = ?$	$DE = ?$

The scale factor from ACB to ADE is $k = \frac{AD}{AC} = \frac{4}{10} = 0,4$ (reduction).

To go to triangle ADE , I multiply by k :

$$DE = 0,4 \times CB = 0,4 \times 7,5 = \boxed{3}$$

$$AE = 0,4 \times AB = 0,4 \times 8 = \boxed{3,2}$$

Check: $\frac{4}{10} = \frac{3,2}{8} = \frac{3}{7,5} = 0,4$: the ratios are equal. The scale factor is strictly less than 1, so each side of ADE must be strictly shorter than its corresponding side in ACB : this is the case, $4 < 10$, $3,2 < 8$ and $3 < 7,5$. ✓

Warning! In e), $[AD]$ lies along $[AB]$ on the diagram, but its corresponding side is $[AC]$: the pairing follows the corresponding vertices ($D \leftrightarrow C$), not the supporting lines. Naïm, in exercise 5.1, makes the same mistake: he matches sides “by eye”.

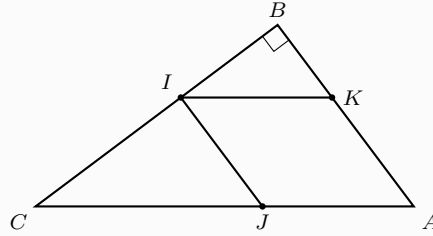
Method. Associated models: **J2** (question a)); **D2** (questions c) and d)); **L1** (questions c) and d)); **A2** (question e)).

a) In triangle ABC , the longest side is $[CA]$.

On the one hand, $CA^2 = 5^2 = 25$. On the other hand, $AB^2 + BC^2 = 3^2 + 4^2 = 9 + 16 = 25$.

Thus $CA^2 = AB^2 + BC^2$. Now, if the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then this triangle is right-angled, and the right angle is opposite the longest side (converse of Pythagoras' theorem). Therefore triangle ABC is right-angled at B . ■

b) I draw the line segment $[CA]$ of length 5, then I use a pair of compasses to plot point B such that $AB = 3$ and $BC = 4$. I plot J on $[CA]$ such that $CJ = 3$. The line parallel to (AB) passing through J intersects (BC) at I ; the line parallel to (CA) passing through I intersects (AB) at K .



$CA = 5, AB = 3, BC = 4$
 $J \in [CA], CJ = 3$
 $(JI) \parallel (AB), (IK) \parallel (CA)$
 ABC right-angled at B

Warning! In b), the figure is drawn full size, with ruler and compasses, using the chosen unit: here, one unit is represented by 1 cm.

c) The angle at C is shared by triangles JIC and ABC (with $J \in [CA]$ and $I \in [CB]$), so $\widehat{JCI} = \widehat{ACB}$.

The parallel lines (JI) and (AB) are cut by the transversal (CA) at J and A respectively. The corresponding angles $\widehat{CJI}$ and $\widehat{CAB}$ therefore have the same measure: $\widehat{CJI} = \widehat{CAB}$.

Now, if two triangles have two pairs of equal angles, then they are similar.

Therefore triangles JIC and ABC are similar, the corresponding vertices being $J \leftrightarrow A, I \leftrightarrow B$ and $C \leftrightarrow C$. ■

From the above, $J \leftrightarrow A, I \leftrightarrow B$ and $C \leftrightarrow C$: the corresponding sides are $[JI] \leftrightarrow [AB], [JC] \leftrightarrow [AC]$ and $[IC] \leftrightarrow [BC]$.

I arrange the corresponding sides in a table:

sides of ABC	$AB = 3$	$AC = 5$	$BC = 4$
sides of JIC	$JI = ?$	$JC = 3$	$CI = ?$

The scale factor from ABC to JIC is $k = \frac{JC}{AC} = \frac{3}{5} = \boxed{0,6}$ (reduction).

To go to triangle JIC , I multiply by k :

$$CI = 0,6 \times BC = 0,6 \times 4 = \boxed{2,4}$$

$$JI = 0,6 \times AB = 0,6 \times 3 = \boxed{1,8}$$

Check: $\frac{1,8}{3} = \frac{3}{5} = \frac{2,4}{4} = 0,6$: the ratios are equal. The scale factor is strictly less than 1, so each side of JIC must be strictly shorter than its corresponding side in ABC : this is the case, $1,8 < 3, 3 < 5$ and $2,4 < 4$. ✓

d) Point I lies on the line segment $[CB]$, so $BI = CB - CI = 4 - 2,4 = \boxed{1,6}$.

The angle at B is shared by triangles KBI and ABC (with $K \in [BA]$ and $I \in [BC]$), so $\widehat{KBI} = \widehat{ABC}$.

The parallel lines (IK) and (CA) are cut by the transversal (BC) at I and C respectively. The corresponding angles $\widehat{BIK}$ and $\widehat{BCA}$ therefore have the same measure: $\widehat{BIK} = \widehat{BCA}$.

Now, if two triangles have two pairs of equal angles, then they are similar.

Therefore triangles KBI and ABC are similar, the corresponding vertices being $K \leftrightarrow A, B \leftrightarrow B$ and $I \leftrightarrow C$. ■

From the above, $K \leftrightarrow A, B \leftrightarrow B$ and $I \leftrightarrow C$: the corresponding sides are $[KB] \leftrightarrow [AB], [KI] \leftrightarrow [AC]$ and $[BI] \leftrightarrow [BC]$.

I arrange the corresponding sides in a table:

sides of ABC	$AB = 3$	$AC = 5$	$BC = 4$
sides of KBI	$KB = ?$	$KI = ?$	$BI = 1,6$

The scale factor from ABC to KBI is $k = \frac{BI}{BC} = \frac{1,6}{4} = \frac{16}{40} = 0,4$ (reduction).

To go to triangle KBI , I multiply by k :

$$KB = 0,4 \times AB = 0,4 \times 3 = \boxed{1,2}$$

$$KI = 0,4 \times AC = 0,4 \times 5 = \boxed{2}$$

Check: $\frac{1,2}{3} = \frac{2}{5} = \frac{1,6}{4} = 0,4$: the ratios are equal. The scale factor is strictly less than 1, so each side of KBI must be strictly shorter than its corresponding side in ABC : this is the case, $1,2 < 3, 2 < 5$ and $1,6 < 4$. ✓

e) Areas are expressed in square units.

Triangle ABC is right-angled at B : its area is $\frac{AB \times BC}{2} = \frac{3 \times 4}{2} = \boxed{6}$.

From question d), from ABC to KBI , the lengths are multiplied by 0,4.

I am looking for an area: it is multiplied by $0,4^2 = 0,16$.

$6 \times 0,16 = 0,96$.

Conclusion: the area of triangle KBI is $\boxed{0,96}$.

Check: $0,96 \div 0,16 = 6$. ✓

From question c), from ABC to JIC , the lengths are multiplied by 0,6.

I am looking for an area: it is multiplied by $0,6^2 = 0,36$.

$6 \times 0,36 = 2,16$.

Conclusion: the area of triangle JIC is $\boxed{2,16}$.

Check: $2,16 \div 0,36 = 6$. ✓

Triangle ABC is made up of triangle JIC , quadrilateral $AKIJ$ and triangle KBI : the area of quadrilateral $AKIJ$ is $6 - 0,96 - 2,16 = \boxed{2,88}$.

Check: quadrilateral $AKIJ$ has parallel opposite sides, $(JI) \parallel (AK)$ and $(IK) \parallel (JA)$: it is a parallelogram. Its base $[AK]$ measures $AB - KB = 3 - 1,2 = 1,8$, and the corresponding height is the distance from I to the line (AB) , that is $IB = 1,6$, because $(BC) \perp (AB)$: its area is $1,8 \times 1,6 = 2,88$. ✓

2.5. The bow-tie with proportional sides

★★★☆

Method. Associated models: **D3** (question a)); **A2** (question b)).

a) The angles $\widehat{DAC}$ and $\widehat{BAE}$ are vertically opposite (with the lines (DE) and (BC) intersecting at A), so $\widehat{DAC} = \widehat{BAE}$.

The sides enclosing these angles measure $AD = 21$ mm and $AC = 72$ mm in triangle DAC , $AB = 28$ mm and $AE = 96$ mm in triangle BAE :

$$\frac{AB}{AD} = \frac{28}{21} = \frac{4}{3} \quad \frac{AE}{AC} = \frac{96}{72} = \frac{4}{3}$$

Both ratios are equal.

Now, if two triangles have an equal angle included between two pairs of proportional sides, then they are similar.

Therefore triangles DAC and BAE are similar, with scale factor $k = \frac{4}{3}$ from DAC to BAE (enlargement), the corresponding vertices being $D \leftrightarrow B$, $A \leftrightarrow A$ and $C \leftrightarrow E$. ■

The criterion used is the third criterion. Only one pair of equal angles is known, and no parallel lines make it possible to obtain a second one: the angle criterion cannot conclude on its own.

b) From question a), from DAC to BAE , the lengths are multiplied by $\frac{4}{3}$.

I am looking for an area: it is multiplied by $\left(\frac{4}{3}\right)^2 = \frac{16}{9}$.

Conclusion: the ratio of the areas, from DAC to BAE , is $\boxed{\frac{16}{9}}$; now $\frac{16}{9} > 1$, so triangle BAE has the larger area.

c) No.

I assume that the lines (DC) and (BE) are parallel.

Points D and E lie on either side of the line (BC) , because the line segment $[DE]$ intersects this line at A : for the transversal (BC) , the angles $\widehat{DCB}$ and $\widehat{EBC}$ are alternate interior angles. Now, if two parallel lines are cut by a transversal, then the alternate interior angles they form are equal. Therefore $\widehat{DCB} = \widehat{EBC}$. Point A lies between B and C , so $\widehat{DCB} = \widehat{DCA}$ and $\widehat{EBC} = \widehat{EBA}$ (they are the same angles), hence $\widehat{DCA} = \widehat{EBA}$.

From question a), $C \leftrightarrow E$. Now, if two triangles are similar, then their corresponding angles are equal. Therefore $\widehat{DCA} = \widehat{BEA}$.

From both equalities, I deduce $\widehat{EBA} = \widehat{BEA}$: in triangle ABE , the angles at B and at E are equal. Now, if a triangle has two equal angles, then it is isosceles, and its two sides of equal length meet at the third vertex. Therefore I obtain $AB = AE$. This is impossible, because $AB = 28$ mm and $AE = 96$ mm: the assumption is false, and the lines (DC) and (BE) are $\boxed{\text{not parallel}}$. ■

To remember. In exercise 1.4, the parallel wings of the bow-tie were given and provided the angles; here, it is the lengths that give the similarity, and the wings are not parallel. The reasoning in c) ("I assume that...", followed by a contradiction) is called a proof by contradiction.

2.5. Optional extension

Detailed solution: why must the angle be included?

A1. Using the order of the points

Since C is between A and H , we have $AH = AC + CH$. Therefore $AC = 4 - 3 = 1$ cm. Since H is between A and D , we have $AD = AH + HD = 4 + 3 = 7$ cm. The given order avoids choosing an addition or a subtraction based on the appearance of the drawing.

A2. Calculating the three missing lengths

Triangles BHC and BHD are right-angled at H , because C, H, D lie on the line (AD) and $(BH) \perp (AD)$. In each one, Pythagoras' theorem gives:

$$BC^2 = BH^2 + HC^2 = 4^2 + 3^2 = 25, \quad BD^2 = BH^2 + HD^2 = 4^2 + 3^2 = 25.$$

Since lengths are positive, $BC = BD = 5$ cm. Triangle BHA is also right-angled at H , so

$$AB^2 = AH^2 + BH^2 = 4^2 + 4^2 = 32, \quad AB = \sqrt{32} \text{ cm.}$$

Since $5^2 = 25 < 32 < 36 = 6^2$, we obtain $5 < \sqrt{32} < 6$. This exact bound will be enough to compare the sides; no decimal approximation is needed.

A3. Justifying the equality of the angles

Triangle ABH is right-angled at H and isosceles, because $AH = BH$. Its other two angles are equal and their sum is 90° : each therefore measures 45° . The points C, H, D lie on the same ray starting at A : the angles $\widehat{BAC}$, $\widehat{BAH}$ and $\widehat{BAD}$ are the same angle. They all measure 45° .

B1. Ordering the sides in the same way

For ABC , we have $1 < 5 < \sqrt{32}$. For ABD , we have $5 < \sqrt{32} < 7$, since $\sqrt{32} < 6 < 7$.

Triangle	Shortest side	Middle side	Longest side
ABC	$AC = 1$	$BC = 5$	$AB = \sqrt{32}$
ABD	$BD = 5$	$AB = \sqrt{32}$	$AD = 7$

The numbers are the lengths in centimetres. The side $[AB]$, present in both triangles, does not have the same rank: it is the longest in the first and the middle side in the second. One must therefore not automatically match a side with itself.

B2. Showing that the sides are not proportional

If the triangles were similar, the shortest side of the first would correspond to the shortest side of the second. The scale factor from ABC to ABD would therefore be $5/1 = 5$ (enlargement). However, the middle sides give $\sqrt{32}/5$. We know that $5 < \sqrt{32} < 6$; dividing by the positive number 5:

$$1 < \frac{\sqrt{32}}{5} < \frac{6}{5} = 1.2.$$

This quotient cannot be equal to 5. Two quotients are different: the side lengths are not proportional. Triangles ABC and ABD are therefore not similar.

B3. Identifying the missing hypothesis precisely

We do have $AB/AB = 1$ and $BC/BD = 5/5 = 1$: the two chosen pairs of sides are proportional. We also have $\widehat{BAC} = \widehat{BAD} = 45^\circ$. However, the sides $[AB]$ and $[BC]$ meet at B , as do $[AB]$ and $[BD]$. The known angle is at A , not at B : it is not included between the compared sides.

The figure disproves the statement: "Two triangles that have two pairs of proportional sides and one pair of equal angles are necessarily similar." All these conditions are satisfied, yet the conclusion was disproved in B2. This does not contradict the true SAS criterion, one hypothesis of which is missing here.

C1. Fixing the criterion

To compare AB with AB and BC with BD , we would need the equality

$$\widehat{ABC} = \widehat{ABD}.$$

These are the angles included between the chosen sides. With this condition, the SAS criterion would ensure similarity with scale factor 1; the triangles would then be congruent. In our figure, this equality is not true: in particular, it would lead to $AC = AD$, whereas $1 \neq 7$.

C2. Reusing the criterion with the right conditions

In the new situation,

$$\frac{EF}{AB} = \frac{FG}{BC} = 2 \quad \text{and} \quad \widehat{EFG} = \widehat{ABC}.$$

The angle at F is indeed included between $[EF]$ and $[FG]$, just as the angle at B is between $[AB]$ and $[BC]$. The SAS similarity criterion, accepted in section 2.3 of the course notes, applies. The correspondences are $A \leftrightarrow E$, $B \leftrightarrow F$, $C \leftrightarrow G$. The side $[EG]$ therefore corresponds to $[AC]$:

$$EG = 2AC = 2 \times 1 = 2 \text{ cm.}$$

The scale factor 2 applies to all three sides. The decisive difference with the counterexample is the position of the known angle, and not whether the scale factor is 1 or 2.

2.6. The shared angle, the midpoint and the areas

★★★☆

Method. Same family as exercises 2.3 (configuration 2) and 6.9: shared angle + marked angle $\rightarrow$ cross-pairing. Two new features: the piece of information “ I midpoint” to be turned into a length before any table, and the conclusion in terms of areas (k^2).



- a) We know that $\widehat{IAD} = \widehat{CAB}$ (shared angle at A , with $I \in [AB]$ and $D \in [AC]$). We likewise know that $\widehat{AID} = \widehat{ACB}$ (marked angle in the problem statement). Now, if two triangles have two pairs of equal angles, then they are similar. Therefore triangles AID and ACB are similar, with $A \leftrightarrow A$, $I \leftrightarrow C$, $D \leftrightarrow B$. ■

Warning! Cross-pairing: I (on $[AB]$) corresponds to C , and D (on $[AC]$) corresponds to B ; the angles decide, not the supporting lines.



- b) I is the midpoint of $[AB]$, so $AI = AB \div 2 = 28 \div 2 = 14$ mm. Corresponding sides: $[AI] \leftrightarrow [AC]$, $[AD] \leftrightarrow [AB]$, $[ID] \leftrightarrow [CB]$. The reduction scale factor from ACB to AID is

$$k = \frac{AI}{AC} = \frac{14}{42} = \frac{1}{3}.$$

Therefore $AD = \frac{1}{3} \times AB = \frac{1}{3} \times 28 = \frac{28}{3}$ mm and $ID = \frac{1}{3} \times CB = \frac{1}{3} \times 39 = \frac{39}{3} = 13$ mm. Check: $\frac{14}{42} = \frac{28/3}{28} = \frac{13}{39} = \frac{1}{3}$ ✓, and

$AD = \frac{28}{3}$ mm $\approx 9,3$ mm < 42 mm: D is indeed inside $[AC]$. ✓

- c) Triangle AID is a reduction of triangle ACB with scale factor $\frac{1}{3}$. Now, in a reduction with scale factor k , areas are multiplied by k^2 . Therefore

$$\frac{\text{area}(AID)}{\text{area}(ABC)} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}. \quad \blacksquare \text{ (Triangles } ACB \text{ and } ABC \text{ are the same triangle: same three vertices.)}$$

3. Enlargements and reductions

3.1. Warm-up drills: the effects of k , k^2 , k^3

★☆☆☆

Method. Associated models: **A2** (questions a), b), c), e) and h)); **A3** (question d)); **A1** (questions f) and g)).

-  a) From triangle ABC to triangle AMN , the lengths are multiplied by 0,25.

I am looking for a perimeter, which is a length: it is multiplied by 0,25.

The perimeter of triangle ABC is $AB + AC + BC = 12 + 8 + 6 = 26$ cm, and $26 \times 0,25 = 6,5$.

Conclusion: the perimeter of triangle AMN is 6,5 cm.

Check: $6,5 \div 0,25 = 26$. ✓

-  b) From the triangle to its enlargement, the lengths are multiplied by 2.

I am looking for an area: it is multiplied by $2^2 = 4$.

$25 \times 4 = 100$.

Conclusion: the new area is 100 cm².

Check: $100 \div 4 = 25$. ✓

- c) From the cube to the scale model, the lengths are multiplied by $\frac{1}{30}$, that is divided by 30.

I am looking for a volume: it is divided by $30^3 = 27\,000$.

$27\,000 \div 27\,000 = 1$.

Conclusion: the volume of the scale model is 1 m³.

Check: $1 \times 27\,000 = 27\,000$. ✓

-  d) The ratio of the areas is 9: this is k^2 .

The positive number whose square is 9 is 3, because $3^2 = 9$.

The scale factor of the enlargement is $k = 3$.

Check: $3^2 = 9$. ✓

- e) The lengths are multiplied by 5.

I am looking for a volume: it is multiplied by $5^3 = 125$.

Conclusion: the volume is multiplied by 125.

-  f) The corresponding lengths are 5,4 cm on the screen and 15 cm on the printed photo; they are in the same unit.

The scale factor from the screen to the printed photo is $k = \frac{15}{5,4} = \frac{150}{54} = \frac{25}{9}$.

We have $k > 1$, so it is an enlargement, with scale factor $k = \frac{25}{9}$.

The exact answer is C.

-  g) A reduction has a scale factor strictly between 0 and 1.

$120\% = 1,2 > 1$ and $\frac{4}{3} > 1$: these are enlargements. $0 < \frac{3}{4} < 1$: this is a reduction.

The exact answer is C.

- h) From the figure to its enlargement, the lengths are multiplied by k . An angle measure is preserved under an enlargement or a reduction.

Conclusion: the angle measures are multiplied by 1: they do not change.

Warning! In f), 2,7 and 2,8 are only approximate values of k : $5,4 \times 2,7 = 14,58$ and $5,4 \times 2,8 = 15,12$, and not 15.

Application 3.1 bis: the photographed pennant

☆☆☆

Method. Associated models: **A1** (question a)); **A2** (question b)).

- a) The corresponding lengths are 4 cm on the photograph and 60 cm on the real pennant; they are in the same unit.

The scale factor from the photograph to the real pennant is $k = \frac{60}{4} = 15$.

We have $k > 1$, so it is an enlargement, with scale factor $k = 15$.

- b) From the photograph to the real pennant, the lengths are multiplied by 15.

I am looking for a length: it is multiplied by 15.

$$3 \times 15 = 45.$$

Conclusion: the 3 cm side on the photo actually measures 45 cm .

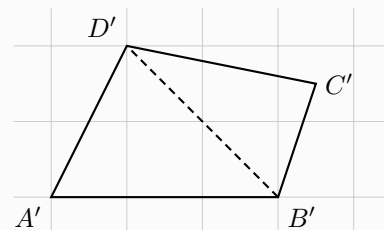
Check: $45 \div 15 = 3$; and $3 < 4$ gives $45 < 60$: the order of the lengths is preserved. ✓

3.2. The quadrilateral: conjecturing with k and k^2

☆☆☆

Method. Associated model: **A2** (questions b) and d)).

- a) I divide each step by 2. Starting from A' , I plot B' 3 squares to the right, C' 3,5 squares to the right and 1,5 squares up, then D' 1 square to the right and 2 squares up. I join these four points in this order, then I draw the diagonal $[B'D']$.



reduction of $ABCD$ with scale factor 0,5;
1 cm grid squares

- b) From $ABCD$ to $A'B'C'D'$, the lengths are multiplied by 0,5.

An angle measure is preserved under an enlargement or a reduction.

Conclusion: the angle $\widehat{D'A'B'}$ has the same measure as the angle $\widehat{DAB}$: $\widehat{D'A'B'} = \widehat{DAB}$.

- c) Triangle ABD has base $[AB]$, of length 6 cm, and corresponding height the distance from D to the line (AB) , read off the grid: 4 cm. Its area is $\frac{6 \times 4}{2} = 12 \text{ cm}^2$.

Triangle $A'B'D'$ has base $[A'B']$, of length 3 cm, and corresponding height the distance from D' to the line $(A'B')$, read off my figure: 2 cm. Its area is $\frac{3 \times 2}{2} = 3 \text{ cm}^2$.

We have $3 = 12 \times 0,25$ and $0,25 = 0,5^2$, so I complete: $\mathcal{A}' = 0,5^2 \times \mathcal{A}$.

- d) From $ABCD$ to $A''B''C''D''$, the lengths are multiplied by 1,2.

I am looking for an area: it is multiplied by $1,2^2 = 1,44$.

From question c), the area of triangle ABD is 12 cm^2 , and $12 \times 1,44 = 17,28$.

Conclusion: the area of triangle $A''B''D''$ is $17,28 \text{ cm}^2$.

Check: $17,28 \div 1,44 = 12$. ✓

- e) "In an enlargement or a reduction with scale factor k , lengths are multiplied by k , angles are preserved, areas are multiplied by k^2 ."

3.3. Working backwards: finding k

☆☆☆

Method. Associated models: **D2** (question a)); **A2** (questions a), c) and d)); **A3** (question b)); **V1** (question d)).

- a) The angle at C is common to triangles CAB and $CA'B'$ (with $A' \in [CA]$ and $B' \in [CB]$), so $\widehat{ACB} = \widehat{A'CB'}$.

The parallel lines (AB) and $(A'B')$ are cut by the transversal (CA) at A and A' respectively. The corresponding angles $\widehat{CAB}$ and $\widehat{CA'B'}$ are therefore equal: $\widehat{CAB} = \widehat{CA'B'}$.

Now, if two triangles have two pairs of equal angles, then they are similar.

Therefore triangles CAB and $CA'B'$ are similar, the corresponding vertices being $C \leftrightarrow C$, $A \leftrightarrow A'$ and $B \leftrightarrow B'$. ■

From CAB to $CA'B'$, the lengths are multiplied by $k = \frac{A'B'}{AB} = \frac{2}{1,5} = \frac{20}{15} = \frac{4}{3}$.

I am looking for an area: it is multiplied by $\left(\frac{4}{3}\right)^2 = \frac{16}{9}$.

$$2,7 \times \frac{16}{9} = \frac{2,7 \times 16}{9} = \frac{43,2}{9} = 4,8.$$

Conclusion: the area of triangle $CA'B'$ is $4,8 \text{ cm}^2$.

Check: $4,8 \div \frac{16}{9} = 4,8 \times \frac{9}{16} = \frac{4,8 \times 9}{16} = \frac{43,2}{16} = 2,7$. ✓

 b) The ratio of the areas is $\frac{1\,800}{72} = 25$: this is k^2 .

The positive number whose square is 25 is 5, because $5^2 = 25$.

The scale factor of the enlargement is $k = 5$.

Check: $72 \times 5^2 = 72 \times 25 = 1\,800$. ✓

 c) From $A'B'C'D'$ to $ABCD$, the lengths are divided by 1,25.

I am looking for an area: it is divided by $1,25^2 = 1,562\,5$.

$24 \div 1,562\,5 = 15,36$.

Conclusion: the area of rectangle $ABCD$ is $15,36 \text{ cm}^2$.

Check: $15,36 \times 1,562\,5 = 24$. ✓

The correct answer is A .

 d) From the plan to the actual pitch, the lengths are multiplied by 3 000.

I am looking for an area: it is multiplied by $3\,000^2 = 9\,000\,000$.

$8 \times 9\,000\,000 = 72\,000\,000 \text{ cm}^2$, and $72\,000\,000 \text{ cm}^2 = 7\,200 \text{ m}^2$, because $1 \text{ m}^2 = 10\,000 \text{ cm}^2$.

Conclusion: the actual pitch has an area of $7\,200 \text{ m}^2$.

Check: $72\,000\,000 \div 9\,000\,000 = 8$. ✓

A football pitch measures about 105 m by 68 m, which is about $7\,100 \text{ m}^2$.

An area of $7\,200 \text{ m}^2$ is very close to this: for a football pitch, the result is plausible.

Warning! In c), the other answers come from dimension or direction errors: $19,2 = 24 \div 1,25$ (dividing by k instead of k^2), $30 = 24 \times 1,25$ (multiplying by k) and $37,5 = 24 \times 1,562\,5$ (multiplying instead of dividing).

3.4. Is Arthur right?

★★★☆☆

Method. Associated models: **D2** (question a)); **A1** (question a)); **A2** (question b)).

 a) The angle at A is common to triangles AMN and ABC (with $M \in [AB]$ and $N \in [AC]$), so $\widehat{MAN} = \widehat{BAC}$.

The parallel lines (MN) and (BC) are cut by the transversal (AB) at M and B respectively. The corresponding angles $\widehat{AMN}$ and $\widehat{ABC}$ are therefore equal: $\widehat{AMN} = \widehat{ABC}$.

Now, if two triangles have two pairs of equal angles, then they are similar.

Therefore triangles AMN and ABC are similar, the corresponding vertices being $A \leftrightarrow A$, $M \leftrightarrow B$ and $N \leftrightarrow C$. ■

The corresponding sides $[AB]$ and $[AM]$ join the corresponding vertices $A \leftrightarrow A$ and $B \leftrightarrow M$.

The scale factor from ABC to AMN is $k = \frac{AM}{AB} = \frac{2}{6} = \frac{1}{3}$.

We have $0 < k < 1$, so triangle AMN is a reduction of triangle ABC , with scale factor $k = \frac{1}{3}$: Arthur is right.

 b) From question a), from ABC to AMN , the lengths are multiplied by $\frac{1}{3}$.

I am looking for an area: it is multiplied by $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$.

$9 \times \frac{1}{9} = \frac{9 \times 1}{9} = \frac{9}{9} = 1$.

Conclusion: the area of triangle AMN is 1 cm^2 .

Check: $1 \div \frac{1}{9} = 1 \times 9 = 9$. ✓

3.5. The Statue of Liberty: a critical look

★★★★☆

Method. Associated models: **A1** (question a)); **V3** (question b)).

 a) The corresponding lengths are 46 m for the New York statue and 11,50 m for the Paris one; they are in the same unit.

The scale factor from the New York statue to the Paris one is $k = \frac{11,5}{46} = \frac{115}{460} = \frac{1}{4} = 0,25$.

We have $0 < k < 1$, so it is a reduction, with scale factor $k = 0,25$.

 b) From the New York statue to the Paris one, the lengths are multiplied by $k = \frac{1}{4}$, as assumed in the problem.

With the same materials in the same places, the mass is proportional to the volume. A volume is multiplied by $k^3 = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$: the Paris statue should have a mass of $225 \times \frac{1}{64} = \frac{225 \times 1}{64} = \frac{225}{64} = 3,515\,625$ t, or about 3,5 t.
The stated mass is 14 t, and $3,515\,625 \neq 14$.

Conclusion: the data are not consistent with a reduction of scale factor $\frac{1}{4}$ keeping the same materials.

Check: $3,515\,625 \times 64 = 225$. ✓

No, I cannot deduce that the outer shapes are not similar: the mass also depends on the inside of the statue, the wall thickness and the materials, which may be different in the replica; the problem does not tell us what has changed.

Warning! The classic mistake: $225 \div 4 = 56,25$ t, which is multiplying the mass by k instead of k^3 .

3.6. Ice cream cones

★★★★

Method. Associated models: **A2** (questions a) and c)); **V3** (question b)).

 a) From the large cone to the medium cone, the lengths are multiplied by $75\% = 0,75$.

I am looking for a volume: it is multiplied by $0,75^3 = 0,421\,875$.

$720 \times 0,421\,875 = 303,75$.

Conclusion: the medium cone has a volume of 303,75 mL.

Check: $303,75 \div 0,421\,875 = 720$. ✓

 b) From the large cone to the small cone, the lengths are multiplied by $k = 50\% = 0,5$.

A volume is multiplied by $k^3 = 0,5^3 = 0,125$: the volume of the small cone is $720 \times 0,125 = 90$ mL.

On the other hand, half the volume of the medium cone is $303,75 \div 2 = 151,875$ mL. Now $90 \neq 151,875$, so this volume is not half that of the medium cone.

Conclusion: Marie is wrong; the small cone has a volume of 90 mL.

Check: $90 \div 0,125 = 720$. ✓

 c) Decreasing by 25% means multiplying by $1 - 0,25 = 0,75$: the lengths are multiplied by 0,75.

I am looking for a volume: it is multiplied by $0,75^3 = 0,421\,875$.

Conclusion: the volume is multiplied by 0,421 875.

The volume therefore decreases by $1 - 0,421\,875 = 0,578\,125$, that is, by 57,812 5 %, or about 58 %.

Moral: a 25 % decrease in lengths causes the volume to decrease by about 58 %: the volume decreases much more than the lengths.

Warning! From the medium cone to the small one, the lengths are multiplied by $\frac{0,5}{0,75} = \frac{2}{3}$, so the volume is multiplied by $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$, about 0,3: much less than half.

3.7. Doubling a cube: the Delian problem

★★☆☆



a) Volume of the starting cube: $V = 10^3 = \span style="border: 1px solid black; padding: 2px;">1\,000\text{ cm}^3.$

b) Cube with edge 20 cm: $V' = 20^3 = \span style="border: 1px solid black; padding: 2px;">8\,000\text{ cm}^3. This is not double: it is eight times the starting volume. Doubling the edge multiplies the volume by $2^3 = 8$. Léa is wrong.$

c) To multiply the volume by 2, we need $k^3 = 2$, so $k = \sqrt[3]{2} \approx 1,26$. The required edge is $10 \times \sqrt[3]{2} \approx 10 \times 1,26 = \span style="border: 1px solid black; padding: 2px;">12,6\text{ cm}.$

d) Check: $12,6^3 = 2\,000,376\text{ cm}^3 \approx 2\,000\text{ cm}^3$, which is indeed about double $1\,000\text{ cm}^3$. ✓

To remember. Doubling a volume does not mean doubling the lengths: the edge is only multiplied by $\sqrt[3]{2} \approx 1,26$ (an enlargement of about 26 %). This is the Greek problem of doubling the cube: impossible to construct with straightedge and compass, but easy to calculate.

3.8. Scratch: two triangles of the same shape

★★☆☆

Method. Associated models: **A1** (questions c) and e)); **R1** (question d)); **A2** (questions d) and e)).

 a) I complete the script: “repeat 3 times” and “turn 120 degrees to the right”.

 b) The triangle closes: the three turns make a full turn, $3 \times 120^\circ = 360^\circ$, and the three sides have the same length, so the sprite returns to its starting point. At each vertex, the sprite turns through the exterior angle; the interior angle is $180^\circ - 120^\circ = 60^\circ$. With 60° turns, three sides do not close the triangle: the sprite only turns through $3 \times 60^\circ = 180^\circ$ in total.

c) The corresponding lengths are 60 steps in the first triangle and 90 steps in the second; they are in the same unit.

The scale factor from the first triangle to the second is $k = \frac{90}{60} = 1,5$.

We have $k > 1$, so it is an enlargement, with scale factor $k = 1,5$.

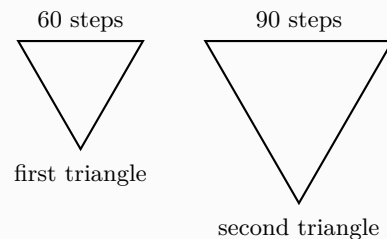
I run the script: with the pen up, the sprite moves to the starting point of the second triangle, points in direction 90° and repeats the same loop; it draws an equilateral triangle of side 90 steps, with no line between the two shapes.

The prediction is right: the sides of the second triangle measure 90 steps, and $1,5 \times 60 = 90$.

d) The first triangle, with sides of 60 steps, is equilateral: its three angles are equal and each measures $180^\circ \div 3 = 60^\circ$. The second triangle, with sides of 90 steps, is equilateral: its three angles are equal and each measures $180^\circ \div 3 = 60^\circ$.

The angles of the first triangle are 60° , 60° and 60° ; those of the second are 60° , 60° and 60° .

The two triangles have pairs of equal angles: by the angle criterion, they are **similar**.



The perimeter of the first triangle is $60 + 60 + 60 = 180$ steps; that of the second is $90 + 90 + 90 = 270$ steps.

From question c), from the first triangle to the second, the lengths are multiplied by 1,5.

I am looking for an area: it is multiplied by $1,5^2 = 2,25$.

Conclusion: the area is multiplied by **2,25**.

e) The corresponding lengths are 60 steps in the first triangle and 30 steps in the second; they are in the same unit.

The scale factor from the first triangle to the second is $k = \frac{30}{60} = 0,5$.

We have $0 < k < 1$, so it is a reduction, with scale factor $k = 0,5$.

From the first triangle to the second, the lengths are multiplied by 0,5.

I am looking for a perimeter, which is a length: it is multiplied by 0,5. An angle measure is preserved under an enlargement or a reduction. I am looking for an area: it is multiplied by $0,5^2 = 0,25$.

$180 \times 0,5 = 90$: the perimeter of the second triangle is 90 steps.

Conclusion: compared with the first triangle, the lengths and perimeter of the second are multiplied by **0,5**, the angle measures by **1** (they do not change) and the area by **0,25**.

Check: $90 \div 0,5 = 180$; when I run the script, I get a triangle with sides of 30 steps, and perimeter $30 + 30 + 30 = 90$ steps. ✓

To remember. Multiplying all lengths by the same number while keeping the angles preserves the shape. Changing the turns does not, in general, give a similar triangle.

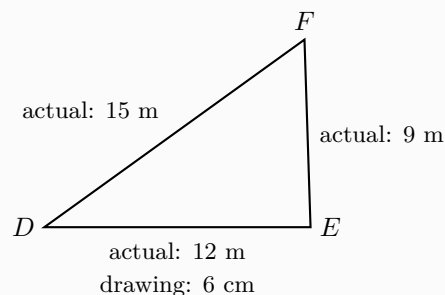
4. Problems

4.1. The triangular park to scale

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Method. Associated models: **A1** (question b)); **A2** (questions c) and e)); **J2** (question d)); **V1** (question e)).

a) I do not mark the right angle: it has not been proved yet.



b) The corresponding lengths are 12 m in reality and 6 cm on the drawing; in the same unit, 12 m = 1 200 cm.

The scale factor from reality to the drawing is $k = \frac{6}{1\,200} = \frac{1}{200}$.

We have $0 < k < 1$, so it is a reduction: the scale of the drawing is $\frac{1}{200}$.

c) From reality to the drawing, the lengths are multiplied by $\frac{1}{200}$, that is, divided by 200.

I am looking for a length: it is divided by 200.

$9 \div 200 = 0,045$ m, and $0,045$ m = 4,5 cm.

Conclusion: on the drawing, EF measures 4,5 cm.

Check: $0,045 \times 200 = 9$. ✓

From reality to the drawing, the lengths are divided by 200.

I am looking for a length: it is divided by 200.

$15 \div 200 = 0,075$ m, and $0,075$ m = 7,5 cm.

Conclusion: on the drawing, DF measures 7,5 cm.

Check: $0,075 \times 200 = 15$. ✓

d) In triangle DEF , the longest side is $[DF]$.

On the one hand, $DF^2 = 15^2 = 225$. On the other hand, $DE^2 + EF^2 = 12^2 + 9^2 = 144 + 81 = 225$.

Thus $DF^2 = DE^2 + EF^2$. Now, if the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then this triangle is right-angled, and the right angle is opposite the longest side (converse of Pythagoras' theorem). Therefore triangle DEF is right-angled at E . ■

Triangle DEF is right-angled at E : its actual area is $\frac{DE \times EF}{2} = \frac{12 \times 9}{2} = \text{span style="border: 1px solid black; padding: 2px;">54 m}^2$.

e) At scale $\frac{1}{200}$, the lengths are divided by 200; an area is divided by $200^2 = 40\,000$.

The area of the drawn triangle is therefore 40 000 times smaller than the actual area, and not 200 times: the answer is no.

From reality to the drawing, the lengths are divided by 200.

I am looking for an area: it is divided by $200^2 = 40\,000$.

$54 \div 40\,000 = 0,001\,35$ m², and $0,001\,35$ m² = 13,5 cm², because 1 m² = 10 000 cm².

Conclusion: the drawn triangle has an area of 13,5 cm².

Check: $0,001\,35 \times 40\,000 = 54$. ✓

Direct calculation gives the same result: the drawn triangle is right-angled at E , like the park, and its area is $\frac{6 \times 4,5}{2} = 13,5$ cm².

4.2. The 1/125 scale model plane

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Method. Associated models: **A2** (questions a), b) and c)); **V1** (question d)).

a) From the plane to the model, the lengths are multiplied by $\frac{1}{125}$, that is, divided by 125.

I am looking for a length: it is divided by 125.

$73 \div 125 = 0,584$ m, and $0,584$ m = 58,4 cm.

Conclusion: the model is 58,4 cm long.

Check: $0,584 \times 125 = 73$. ✓

b) From the model to the plane, the lengths are multiplied by 125.

I am looking for an area: it is multiplied by $125^2 = 15\,625$.

$540,8 \times 15\,625 = 8\,450\,000$ cm², and $8\,450\,000$ cm² = 845 m², because 1 m² = 10 000 cm².

Conclusion: the wings of the plane have an area of 845 m².

Check: $8\,450\,000 \div 15\,625 = 540,8$. ✓

✎ c) From the plane to the model, the lengths are divided by 125.

I am looking for a volume: it is divided by $125^3 = 1\,953\,125$.

$310\,000 \div 1\,953\,125 = 0,158\,72$ L, and $0,158\,72$ L = $158,72$ cm³, because 1 L = $1\,000$ cm³.

Conclusion: the tank of the model has a capacity of $\boxed{158,72 \text{ cm}^3}$.

Check: $0,158\,72 \times 1\,953\,125 = 310\,000$. ✓

✎ d) Question a): An A4 sheet is about 30 cm long.

A 58,4 cm model, roughly two A4 sheets placed end to end, fits on a table: for a model plane, the result is plausible.

Question b): A handball court measures 40 m by 20 m, which is 800 m².

A wing area of 845 m², slightly larger than a handball court, lifts a 73 m long plane: for a very large airliner, the result is plausible.

Question c): A glass of water holds about 200 mL, that is, 200 cm³.

A tank of 158,72 cm³, slightly less than a glass of water, fits inside a 58,4 cm model: for the tank of a model, the result is plausible.

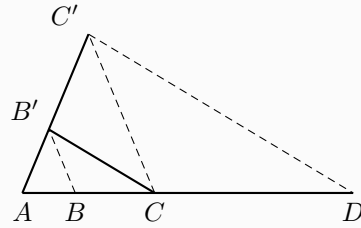
Warning! In b), multiplying by 125 instead of 125^2 would give $540,8 \times 125 = 67\,600$ cm², or 6,76 m²: a wing area of less than 7 m² for a 73 m long plane is implausible.

4.3. The cascade of parallel lines 🏰

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a) Here is a possible construction.



b) In triangles ABB' and ACC' , the angle at A is common ($\widehat{BAB'} = \widehat{CAC'}$), and $\widehat{ABB'} = \widehat{ACC'}$ (corresponding angles formed by the parallel lines (BB') and (CC') cut by the transversal (AC)). Now, if two triangles have two pairs of equal angles, then they are similar. Therefore ABB' and ACC' are similar ($A \leftrightarrow A, B \leftrightarrow C, B' \leftrightarrow C'$), and the lengths of their corresponding sides are proportional:

$$\frac{AB}{AC} = \frac{AB'}{AC'}.$$

c) Similarly, in triangles $AB'C$ and $AC'D$, the angle at A is common ($\widehat{B'AC} = \widehat{C'AD}$), and $\widehat{AB'C} = \widehat{AC'D}$ (corresponding angles formed by the parallel lines $(B'C)$ and $(C'D)$ cut by the transversal (AC')). They are therefore similar ($A \leftrightarrow A, B' \leftrightarrow C', C \leftrightarrow D$), hence

$$\frac{AB'}{AC'} = \frac{AC}{AD}.$$

d) From the two previous equalities, we deduce $\frac{AB}{AC} = \frac{AB'}{AC'} = \frac{AC}{AD}$, so $\boxed{\frac{AB}{AC} = \frac{AC}{AD}}$. ■

e) With $AB = 1$ and $AC = x$, we obtain $\frac{1}{x} = \frac{x}{AD}$, hence (by cross-multiplication) $AD = x \times x = \boxed{x^2}$. To obtain x^3 , we extend the cascade by one step (line parallel to (CC') through D , meeting (AB') at D' , then line parallel to $(C'D)$ through D' , meeting (AC) at E): the same reasoning gives $\frac{AB}{AC} = \frac{AC}{AD} = \frac{AD}{AE}$, so $\frac{1}{x} = \frac{x^2}{AE}$, hence $AE = x \times x^2 = \boxed{x^3}$. The figure “multiplies by x ” at each step.

4.4. The two poles

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Method. Twice the “common angle + right angle” pattern (in nested triangles: the three poles are parallel), and a concluding trick: add the two equalities to bring out $KL + LH = KH$.



- a) The lines (AH) , (CL) and (BK) are all three perpendicular to the line (HK) : the poles are vertical on horizontal ground, and $(CL) \perp (HK)$ by definition of L . Now, if two lines are perpendicular to the same line, then they are parallel to each other. Therefore $(AH) \parallel (CL) \parallel (BK)$. ■
- b) In triangles KLC and KHA , we have $\widehat{LKC} = \widehat{HKA}$ (common angle at K , since $L \in [HK]$ and $C \in [AK]$), and $\widehat{KLC} = \widehat{KHA} = 90^\circ$. Now, if two triangles have two pairs of equal angles, then they are similar. Therefore triangles KLC and KHA are similar ($K \leftrightarrow K, L \leftrightarrow H, C \leftrightarrow A$), hence $\frac{CL}{AH} = \frac{KL}{KH}$, that is, $\frac{CL}{3} = \frac{KL}{KH}$. ■
- c) Similarly, in triangles HLC and HKB , the angle at H is common ($\widehat{LHC} = \widehat{KHB}$) and $\widehat{HLC} = \widehat{HKB} = 90^\circ$: they are therefore similar ($H \leftrightarrow H, L \leftrightarrow K, C \leftrightarrow B$), hence $\frac{CL}{BK} = \frac{HL}{HK}$, that is, $\frac{CL}{2} = \frac{HL}{HK}$. ■
- d) Adding the two equalities:

$$\frac{CL}{3} + \frac{CL}{2} = \frac{KL}{KH} + \frac{HL}{HK} = \frac{KL + LH}{HK} = \frac{HK}{HK} = 1$$

(since L belongs to the segment $[HK]$). Therefore $CL \times \left(\frac{1}{3} + \frac{1}{2}\right) = 1$, so $CL \times \left(\frac{2}{6} + \frac{3}{6}\right) = CL \times \frac{5}{6} = 1$, hence

$$CL = \frac{6}{5} = \boxed{1,20 \text{ m}}.$$

Check: $1,20 < 2 < 3$: the crossing point is indeed lower than both tops ✓, and $\frac{1,20}{3} + \frac{1,20}{2} = 0,4 + 0,6 = 1$. ✓

- e) The distance HK between the poles does not appear in the calculation: the height CL of the crossing point does not depend on it. Moving the poles closer together or further apart does not change CL at all (only the point L moves).

To remember. The superb general result: $\frac{1}{CL} = \frac{1}{AH} + \frac{1}{BK}$ (dividing the equality $\frac{CL}{AH} + \frac{CL}{BK} = 1$ by CL). Physicists will meet it again in Year 11 and in upper secondary school (resistors in parallel, thin lenses): CL is said to be half the harmonic mean of AH and BK .

4.4. Optional extension

Detailed solution: choosing the poles

A1. Reviewing the geometry before calculating

The three lines (AH) , (CL) and (BK) are perpendicular to the ground (HK) ; they are therefore parallel. Triangles KLC and KHA have a common angle at K and a right angle at L and H respectively. They are similar, with $K \leftrightarrow K, L \leftrightarrow H, C \leftrightarrow A$. We obtain

$$\frac{CL}{AH} = \frac{KL}{KH}, \quad \text{that is,} \quad \frac{h}{a} = \frac{KL}{KH}.$$

Similarly, HLC and HKB share the same angle at H and are right-angled at L and K respectively. Their corresponding sides give $h/b = HL/HK$.

The point L lies between H and K , so $KL + LH = KH$. The denominators HK and KH represent the same length. Adding gives:

$$\frac{h}{a} + \frac{h}{b} = \frac{KL + LH}{HK} = 1.$$

The crossing point is above the ground, so $h > 0$: dividing by h is allowed. Since $(h/a)/h = 1/a$, we obtain

$$\boxed{\frac{1}{h} = \frac{1}{a} + \frac{1}{b}}.$$

This relation comes from similar triangles. It does not require knowing the spacing HK .

A2. Checking a first choice

With $a = 3$ and $b = 6$:

$$\frac{1}{h} = \frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.$$

Two positive numbers with the same reciprocal are equal: $h = 2$. The crossing point is therefore at 2 m. Any strictly positive spacing between the bases gives the same height, since the relation depends only on a and b .

A3. A first geometric constraint

In $1/h = 1/a + 1/b$, the term $1/b$ is strictly positive. Thus $1/h > 1/a$. For positive numbers, taking the reciprocal reverses the order: $h < a$. Similarly, $1/a > 0$ implies $1/h > 1/b$, so $h < b$. The crossing point is therefore strictly lower than each of the two tops, as suggested by the figure. Equality with the height of a pole is ruled out as long as the other pole has a finite positive height.

B1. Ruling out poles that are too short

We set $h = 2$. Question A3 gives $2 < a$ and $2 < b$. Since a is an integer, its first possible value is 3. The values 1 and 2 are not suitable; the crossing point cannot be as high as the top of the shorter pole.

B2. Ruling out all large values at once

We chose $a \leq b$. If $a \geq 5$, then $b \geq 5$ as well. Comparing reciprocals:

$$\frac{1}{a} \leq \frac{1}{5}, \quad \frac{1}{b} \leq \frac{1}{5}, \quad \text{so} \quad \frac{1}{a} + \frac{1}{b} \leq \frac{2}{5}.$$

However, we need $1/a + 1/b = 1/2$, and $2/5 < 1/2$ since $4/10 < 5/10$. This is impossible. All values $a \geq 5$ are ruled out by the same reasoning, without having to test them one by one. Together with B1, only $a = 3$ or $a = 4$ remain.

B3. Treating all remaining cases

In the relation $1/a + 1/b = 1/2$, we subtract $1/a$ from each side:

$$\frac{1}{b} = \frac{1}{2} - \frac{1}{a}.$$

The table then gives the reciprocal of b , and then b itself:

a	$1/b$	b	Constraints
3	$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$	6	integer, $6 \geq 3$
4	$\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$	4	integer, $4 \geq 4$

Be sure to invert the last fraction: $1/b = 1/6$ gives $b = 6$, and not $b = 1/6$.

B4. Checking sufficiency and completeness

Both choices work: $1/3 + 1/6 = 1/2$ and $1/4 + 1/4 = 1/2$. For the cables joining the two chosen poles, A1 therefore gives $1/h = 1/2$, so $h = 2$: these are indeed solutions to the geometric problem, not just values taken from a table.

Conversely, any solution can be ordered with $a \leq b$. B1 and B2 then force $a = 3$ or 4 , and B3 determines a unique value of b in each case. There is therefore no other choice:

$$(a; b) = (3; 6) \quad \text{or} \quad (4; 4).$$

The heights are in metres. The choice $(6; 3)$ is the same as $(3; 6)$ with the poles swapped: it is not counted a second time.

C1. Repeating the search with $h = 3$

The condition is now $1/a + 1/b = 1/3$. By A3, $a > 3$, so $a \geq 4$. If $a \geq 7$, then $b \geq 7$ and

$$\frac{1}{a} + \frac{1}{b} \leq \frac{2}{7} < \frac{1}{3} \quad \text{since} \quad \frac{2}{7} = \frac{6}{21} < \frac{7}{21} = \frac{1}{3}.$$

Only $a = 4, 5$ or 6 remain. For each, $1/b = 1/3 - 1/a$:

a	$1/b$	b	Decision
4	$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$	12	suitable: integer and $12 \geq 4$
5	$\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$	$\frac{15}{2} = 7,5$	rejected: non-integer height
6	$\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$	6	suitable: integer and $6 \geq 6$

With poles of 5 m and 7,5 m, the height would indeed be 3 m, but the whole-number height constraint would not be satisfied. All conditions must be checked, not only the equation.

We check $1/4 + 1/12 = 1/3$ and $1/6 + 1/6 = 1/3$: each pair kept indeed gives $h = 3$. The previous bounds and the table have covered all possible values of the smaller pole. The choices are therefore exactly

$$(a; b) = (4; 12) \quad \text{or} \quad (6; 6).$$

C2. Distinguishing two selection criteria

The taller pole measures 12 m with $(4; 12)$, compared with 6 m with $(6; 6)$. To make the taller pole as short as possible, one must choose two 6 m poles.

This is not the choice that uses the smallest possible pole: the latter measures 4 m in the pair $(4; 12)$. Minimising the shorter pole as much as possible here forces us to increase the other. The best choice therefore depends on the chosen criterion, and the complete enumeration makes it possible to justify it.

4.5. A cascade of similarities: altitudes in action

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Method. Associated model: **J1**. Twice the same pattern: “common angle at B + right angle”, the altitude configuration from section 3.3, then an elimination: each of the two similarities expresses the same product $BE \times BA$; by equating them, AB disappears. Carrying out the calculation with letters right to the end is what makes this visible.



- a) In triangles ABC and CBE , we have $\widehat{ABC} = \widehat{CBE}$ (common angle at B , since $E \in [AB]$), and $\widehat{ACB} = \widehat{CEB} = 90^\circ$ (ABC is right-angled at C , and (CE) is the altitude from C so $(CE) \perp (AB)$). Now, if two triangles have two pairs of equal angles, then they are similar. Therefore ABC and CBE are similar, with $A \leftrightarrow C$, $B \leftrightarrow B$, $C \leftrightarrow E$. Hence $\frac{AB}{CB} = \frac{BC}{BE} = \frac{AC}{CE}$, and cross-multiplying on the first equality gives $BC^2 = BE \times BA$. ■
- b) In triangles ABD and FBE , we have $\widehat{ABD} = \widehat{FBE}$ (it is the same angle: $F \in [BD]$ and $E \in [BA]$, with $E \neq B$ and $F \neq B$), and $\widehat{ADB} = \widehat{FEB} = 90^\circ$ (ABD is right-angled at D , and $F \in [CE]$ with $(CE) \perp (AB)$). Now, if two triangles have two pairs of equal angles, then they are similar. Therefore ABD and FBE are similar, with $A \leftrightarrow F$, $B \leftrightarrow B$, $D \leftrightarrow E$. Hence $\frac{AB}{FB} = \frac{BD}{BE}$, and cross-multiplying gives $BF \times BD = BE \times BA$. ■
- c) Questions a) and b) give two expressions for the same product $BE \times BA$:

$$BF \times BD = BC^2 \quad \text{so} \quad BF = \frac{BC^2}{BD} = \frac{4^2}{6} = \frac{16}{6} = \boxed{\frac{8}{3}}.$$

Check: $\frac{8}{3} \approx 2,7 < 6 = BD$: F is indeed between B and D . ✓

- d) Nothing changes: $BF = \frac{BC^2}{BD} = \frac{8}{3}$ does not depend on AB . If AB increases, points E and F move, but BF remains equal to $\frac{8}{3}$: the “expected” piece of data AB was not needed. It is the final algebraic formula that reveals this.

5. Justifying and reasoning

5.1. Spotting the error: Naïm’s work

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Method. Associated models: **V2** (question a)); **L1** (question b)).

- a) I read through the work in the order of the steps: Naïm calculates a ratio without having matched the sides.

The error is at the “matching” step: Naïm paired BC and DE because they both measure 6, but they are not corresponding sides. Yet corresponding sides join corresponding vertices: with $A \leftrightarrow D$ and $B \leftrightarrow E$, the side corresponding to $[DE]$ is $[AB]$, and the one corresponding to $[BC]$ is $[EF]$.

- b) The angles give $A \leftrightarrow D$ and $B \leftrightarrow E$, and therefore also $C \leftrightarrow F$, the third angles: the corresponding sides are $[AB] \leftrightarrow [DE]$, $[AC] \leftrightarrow [DF]$ and $[BC] \leftrightarrow [EF]$.

I arrange the corresponding sides in a table:

sides of ABC	$AB = 4$	AC	$BC = 6$
sides of DEF	$DE = 6$	DF	$EF = ?$

The scale factor from ABC to DEF is $k = \frac{DE}{AB} = \frac{6}{4} = 1,5$ (enlargement).

To go to triangle DEF , I multiply by k : $EF = 1,5 \times BC = 1,5 \times 6 = \boxed{9}$.

Check: $\frac{6}{4} = \frac{9}{6} = 1,5$: the ratios are equal. The scale factor is strictly greater than 1, so each side of DEF must be strictly longer than its corresponding side in ABC : this is the case, $6 > 4$ and $9 > 6$. ✓

Warning! Naïm also concludes that the triangles are congruent. Two similar triangles are congruent only if their scale factor is 1; here, it is 1,5.

5.2. Checking consistency: without calculating

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Method. Associated model: **V1**.

- a) In an enlargement with scale factor 3, areas are multiplied by $3^2 = 9$. The area cannot therefore have been multiplied by 6: the statement is false.

- b) In a reduction, the scale factor is strictly between 0 and 1: all lengths become shorter.

An 8 cm side cannot therefore become a longer 16 cm side: the statement is false.

- c) In a triangle, the longest side is opposite the largest angle; in a triangle that is not equilateral, the largest angle is strictly greater than the smallest.

In the second triangle, the longest side cannot therefore be opposite the smallest angle: the statement is false.

- d) At a scale of $\frac{1}{100}$, lengths are divided by 100; a surface area is an area, so it is divided by $100^2 = 10\,000$.

The painted area of the scale model is therefore 10 000 times smaller than that of the real object, and not 100 times: the statement is false.

5.3. Writing a complete solution

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Method. Associated models: **D2** (question a)); **L1** (questions b) and c), then 1. and 3. in the box).

a) The angles $\widehat{AOB}$ and $\widehat{COD}$ are vertically opposite (with lines (AC) and (BD) intersecting at O), so $\widehat{AOB} = \widehat{COD}$.

From the problem statement, we know that $\widehat{OAB} = \widehat{OCD}$.

Now, if two triangles have two pairs of equal angles, then they are similar.

Therefore triangles OAB and OCD are similar, the corresponding vertices being $O \leftrightarrow O$, $A \leftrightarrow C$ and $B \leftrightarrow D$. ■

b) From question a), $O \leftrightarrow O$, $A \leftrightarrow C$ and $B \leftrightarrow D$: the corresponding sides are $[OA] \leftrightarrow [OC]$, $[OB] \leftrightarrow [OD]$ and $[AB] \leftrightarrow [CD]$.

I arrange the corresponding sides in a table:

sides of OAB	$OA = 6$	$OB = 5,4$	$AB = ?$
sides of OCD	$OC = 4$	$OD = ?$	$CD = 5$

The scale factor from OAB to OCD is $k = \frac{OC}{OA} = \frac{4}{6} = \frac{2}{3}$ (reduction).

To go to triangle OCD , I multiply by k : $OD = \frac{2}{3} \times OB = \frac{2}{3} \times 5,4 = \frac{2 \times 5,4}{3} = \frac{10,8}{3} = 3,6$.

To go back to triangle OAB , I divide by k ; dividing by $\frac{2}{3}$ is multiplying by its reciprocal $\frac{3}{2}$: $AB = CD \div \frac{2}{3} = 5 \times \frac{3}{2} = \frac{5 \times 3}{2} = \frac{15}{2} = 7,5$.

c) Check: $\frac{4}{6} = \frac{3,6}{5,4} = \frac{5}{7,5} = \frac{2}{3}$: the ratios are equal. The scale factor is strictly less than 1, so each side of OCD must be strictly shorter than its corresponding side in OAB : this is the case, $4 < 6$, $3,6 < 5,4$ and $5 < 7,5$. ✓

Conclusion: triangles OAB and OCD are similar, with scale factor $\frac{2}{3}$ from OAB to OCD (reduction), and the required lengths are $OD = 3,6$ and $AB = 7,5$.

1. With this value, the scale factor $k = \frac{OC}{OA}$ becomes smaller, because OC goes from 4 to 3 and OA does not change: I predict that the lengths obtained by multiplying by k decrease, and that those obtained by dividing by k increase. Here, $OD = k \times OB$ decreases and $AB = CD \div k$ increases. I check:

The proof in question a) uses no lengths: with $OC = 3$, triangles OAB and OCD remain similar, with the same corresponding sides.

I arrange the corresponding sides in a table, with $OC = 3$:

sides of OAB	$OA = 6$	$OB = 5,4$	$AB = ?$
sides of OCD	$OC = 3$	$OD = ?$	$CD = 5$

The scale factor from OAB to OCD is $k = \frac{OC}{OA} = \frac{3}{6} = 0,5$ (reduction).

To go to triangle OCD , I multiply by k : $OD = 0,5 \times OB = 0,5 \times 5,4 = 2,7$.

To go back to triangle OAB , I divide by k : $AB = CD \div 0,5 = 5 \div 0,5 = 10$.

Check: $\frac{3}{6} = \frac{2,7}{5,4} = \frac{5}{10} = 0,5$: the ratios are equal. The scale factor is strictly less than 1, so each side of OCD must be strictly shorter than its corresponding side in OAB : this is the case, $3 < 6$, $2,7 < 5,4$ and $5 < 10$. ✓

The prediction is correct: k goes from $\frac{2}{3}$ to $0,5$, OD decreases from $3,6$ to $2,7$ and AB increases from $7,5$ to 10 .

2. Problem: "Triangles ABC and DEF are similar, with $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$. Given $AB = 8$ cm, $BC = 6$ cm, $DE = 6$ cm and $DF = 7,5$ cm, calculate EF and AC ."

The constraints are met: $\frac{DE}{AB} = \frac{6}{8} = \frac{3}{4}$, so DEF is a reduction of ABC with scale factor $\frac{3}{4}$; the unknown length AC is in triangle ABC , and EF is in triangle DEF ; the vertex correspondence is stated.

3. From the problem statement, $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$: the corresponding sides are $[AB] \leftrightarrow [DE]$, $[AC] \leftrightarrow [DF]$ and $[BC] \leftrightarrow [EF]$.

I arrange the corresponding sides in a table:

sides of ABC	$AB = 8$	$AC = ?$	$BC = 6$
sides of DEF	$DE = 6$	$DF = 7,5$	$EF = ?$

The scale factor from ABC to DEF is $k = \frac{DE}{AB} = \frac{6}{8} = 0,75$ (reduction).

To go to triangle DEF , I multiply by k : $EF = 0,75 \times BC = 0,75 \times 6 = 4,5$ cm.

To go back to triangle ABC , I divide by k : $AC = DF \div 0,75 = 7,5 \div 0,75 = 10$ cm.

Check: $\frac{6}{8} = \frac{7,5}{10} = \frac{4,5}{6} = 0,75$: the ratios are equal. The scale factor is strictly less than 1, so each side of DEF must be strictly shorter than its corresponding side in ABC : this is the case, $6 < 8$, $7,5 < 10$ and $4,5 < 6$. ✓

Before giving the problem to a classmate, I check that it gives the vertex correspondence and all the needed lengths: this is the case, and each of the four lengths was used.

5.4. Proving perpendicularity: the two midpoints of the square

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Method. Associated model: **J1**. A cascading game plan: congruent triangles (the square data) $\rightarrow$ equal angles $\rightarrow$ similar triangles $\rightarrow$ the angle at H is 90° . As in the proof in the course notes and in 7.5, congruence and similarity are used in the same proof.



- a) In triangles ABI and DAJ , we have: $AB = DA$ (sides of the square); $\widehat{BAI} = \widehat{ADJ} = 90^\circ$ (angles of the square); $AI = DJ$ (halves of sides $[AD]$ and $[DC]$, which have the same length). The right angle is included between the two stated sides: by the “side-angle-side” congruence criterion, triangles ABI and DAJ are congruent, with $A \leftrightarrow D$, $B \leftrightarrow A$ and $I \leftrightarrow J$. ■
- b) Now, if two triangles are congruent, then their corresponding angles are equal: therefore $\widehat{ABI} = \widehat{DAJ}$ and $\widehat{AIB} = \widehat{DJA}$. ■
- c) On the one hand, $\widehat{ABH} = \widehat{ABI} = \widehat{DAJ}$ (question b, and $H \in [BI]$). On the other hand, $\widehat{HAB} = \widehat{JAB} = \widehat{DAB} - \widehat{DAJ} = 90^\circ - \widehat{DAJ}$; now, in triangle DAJ right-angled at D , we also have $\widehat{DJA} = 90^\circ - \widehat{DAJ}$; therefore $\widehat{HAB} = \widehat{DJA}$. Triangles AHB and JDA have two pairs of equal angles: they are similar, with $A \leftrightarrow J$, $H \leftrightarrow D$, $B \leftrightarrow A$. ■
- d) Now, if two triangles are similar, then their corresponding angles are equal: therefore $\widehat{AHB} = \widehat{JDA} = 90^\circ$. Thus $(AJ) \perp (BI)$. ■
- e) Alternative method. In triangle ABH : $\widehat{HAB} + \widehat{ABH} = (90^\circ - \widehat{DAJ}) + \widehat{DAJ} = 90^\circ$, so $\widehat{AHB} = 180^\circ - 90^\circ = 90^\circ$. Comparison. Method e) is shorter; but method c)–d) additionally proves that AHB and JDA are similar, giving all the length ratios ($\frac{AH}{JD} = \frac{HB}{DA} = \frac{AB}{JA}$) for free: it is the one that generalises as soon as we want to calculate lengths in the figure.

To remember. To prove perpendicularity without measurements: aim for a 90° angle carried over by congruent or similar triangles. And remember the square pair: the “vertex $\rightarrow$ midpoint” segments $[AJ]$ and $[BI]$ are always perpendicular (and of the same length!): true in every square, whatever its size.

6. Brevet-style problems (French national exam at the end of Year 10)

6.1. Brevet 1: Similar triangles on a line

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Method. Associated models: **D1** (question a)); **D2** (question b)); **L1** (question c)).

a) We know that lines (SU) and (RV) are parallel, and that line (ST) intersects them at S and R ; angles $\widehat{USR}$ and $\widehat{VRT}$ are corresponding angles. Now, if two parallel lines are cut by a transversal, then the corresponding angles they form are equal. Therefore $\widehat{USR} = \widehat{VRT}$. ■

b) We know that $\widehat{USR} = \widehat{VRT}$, from question a). Similarly, the parallel lines (RU) and (TV) are cut by the transversal (ST) at R and T respectively. The corresponding angles $\widehat{URS}$ and $\widehat{VTR}$ are therefore equal: $\widehat{URS} = \widehat{VTR}$. Now, if two triangles have two pairs of equal angles, then they are similar. Therefore triangles SRU and RTV are similar, the corresponding vertices being $S \leftrightarrow R$, $R \leftrightarrow T$ and $U \leftrightarrow V$. ■

c) From question b), $S \leftrightarrow R$, $R \leftrightarrow T$ and $U \leftrightarrow V$: the corresponding sides are $[SR] \leftrightarrow [RT]$, $[SU] \leftrightarrow [RV]$ and $[RU] \leftrightarrow [TV]$. Point R lies on segment $[ST]$, so $RT = ST - SR = 10 - 4 = 6$. I arrange the corresponding sides in a table:

sides of RTV	$RT = 6$	$RV = 5,4$	$TV = 7,2$
sides of SRU	$SR = 4$	$SU = ?$	$RU = ?$

The scale factor from RTV to SRU is $k = \frac{SR}{RT} = \frac{4}{6} = \frac{2}{3}$ (reduction).

To go to triangle SRU , I multiply by k :

$$SU = \frac{2}{3} \times RV = \frac{2}{3} \times 5,4 = \frac{2 \times 5,4}{3} = \frac{10,8}{3} = 3,6$$

$$RU = \frac{2}{3} \times TV = \frac{2}{3} \times 7,2 = \frac{2 \times 7,2}{3} = \frac{14,4}{3} = 4,8$$

Check: $\frac{4}{6} = \frac{3,6}{5,4} = \frac{4,8}{7,2} = \frac{2}{3}$: the ratios are equal. The scale factor is strictly less than 1, so each side of SRU must be strictly shorter than its corresponding side in RTV : this is the case, $4 < 6$, $3,6 < 5,4$ and $4,8 < 7,2$. ✓

6.2. Brevet 2: The butterfly

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Method. Associated models: **D2** (question a)); **L1** (question b)).

a) The angles $\widehat{AOB}$ and $\widehat{COD}$ are vertically opposite (with lines (AC) and (BD) intersecting at O), so $\widehat{AOB} = \widehat{COD}$. The parallel lines (AB) and (CD) are cut by the transversal (AC) at A and C respectively. The alternate interior angles $\widehat{OAB}$ and $\widehat{OCD}$ are therefore equal: $\widehat{OAB} = \widehat{OCD}$. Now, if two triangles have two pairs of equal angles, then they are similar. Therefore triangles OAB and OCD are similar, the corresponding vertices being $O \leftrightarrow O$, $A \leftrightarrow C$ and $B \leftrightarrow D$. ■

 b) From question a), $O \leftrightarrow O$, $A \leftrightarrow C$ and $B \leftrightarrow D$: the corresponding sides are $[OA] \leftrightarrow [OC]$, $[OB] \leftrightarrow [OD]$ and $[AB] \leftrightarrow [CD]$.
I arrange the corresponding sides in a table:

sides of OAB	$OA = 3$	$OB = 2,4$	$AB = 4,2$
sides of $ OCD$	$OC = 5$	$OD = ?$	$CD = ?$

The scale factor from OAB to OCD is $k = \frac{OC}{OA} = \frac{5}{3}$ (enlargement).

To go to triangle OCD , I multiply by k :

$$CD = \frac{5}{3} \times AB = \frac{5}{3} \times 4,2 = \frac{5 \times 4,2}{3} = \frac{21}{3} = \boxed{7}$$

$$OD = \frac{5}{3} \times OB = \frac{5}{3} \times 2,4 = \frac{5 \times 2,4}{3} = \frac{12}{3} = \boxed{4}$$

Check: $\frac{5}{3} = \frac{4}{2,4} = \frac{7}{4,2}$: the ratios are equal. The scale factor is strictly greater than 1, so each side of OCD must be strictly longer than its corresponding side in OAB : this is the case, $5 > 3$, $4 > 2,4$ and $7 > 4,2$. ✓

6.3. Brevet 3: Similar triangles and areas

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Method. Associated models: **A1** (question a)); **A2** (questions b) and c)).

 a) The corresponding sides $[AB]$ and $[DE]$ join the corresponding vertices $A \leftrightarrow D$ and $B \leftrightarrow E$.

The scale factor from ABC to DEF is $k = \frac{DE}{AB} = \frac{4}{6} = \frac{2}{3}$.

We have $0 < k < 1$, so triangle DEF is a reduction of triangle ABC , with scale factor $k = \frac{2}{3}$.

 b) From question a), from ABC to DEF , lengths are multiplied by $\frac{2}{3}$.

I am looking for a perimeter, which is a length: it is multiplied by $\frac{2}{3}$.

$$18 \times \frac{2}{3} = \frac{18 \times 2}{3} = \frac{36}{3} = 12.$$

Conclusion: the perimeter of triangle DEF is $\boxed{12 \text{ cm}}$.

Check: $12 \div \frac{2}{3} = 12 \times \frac{3}{2} = \frac{12 \times 3}{2} = \frac{36}{2} = 18$. ✓

 c) From question a), from ABC to DEF , lengths are multiplied by $\frac{2}{3}$.

I am looking for an area: it is multiplied by $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$.

$$13,5 \times \frac{4}{9} = \frac{13,5 \times 4}{9} = \frac{54}{9} = 6.$$

Conclusion: the area of triangle DEF is $\boxed{6 \text{ cm}^2}$.

Check: $6 \div \frac{4}{9} = 6 \times \frac{9}{4} = \frac{6 \times 9}{4} = \frac{54}{4} = 13,5$. ✓

6.4. Brevet 4: Recognise, then calculate

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Method. Associated models: **R1** (question a)); **L1** (questions b) and c)).

 a) Triangle JKL : $\widehat{L} = 180^\circ - 35^\circ - 60^\circ = 85^\circ$. Triangle MNP : $\widehat{P} = 180^\circ - 85^\circ - 60^\circ = 35^\circ$.

The angles of triangle JKL are 35° , 60° and 85° ; those of triangle MNP are 35° , 60° and 85° .

The two triangles have pairs of equal angles: by the angle criterion, they are $\boxed{\text{similar}}$.

 b) The corresponding vertices are those of the equal angles: $J \leftrightarrow P$ (35°), $K \leftrightarrow N$ (60°) and $L \leftrightarrow M$ (85°).

 c) From question b), $J \leftrightarrow P$, $K \leftrightarrow N$ and $L \leftrightarrow M$: the corresponding sides are $[JK] \leftrightarrow [PN]$, $[JL] \leftrightarrow [PM]$ and $[KL] \leftrightarrow [NM]$.

I arrange the corresponding sides in a table:

sides of $ JKL$	$JK = 8$	JL	$KL \approx 4,6$
sides of $ PNM$	$PN = 6$	PM	$NM = ?$

The scale factor from JKL to PNM is $k = \frac{PN}{JK} = \frac{6}{8} = 0,75$ (reduction).

To go to triangle PNM , I multiply by k : $NM = 0,75 \times KL \approx 0,75 \times 4,6 = 3,45$, so $\boxed{NM \approx 3,45 \text{ cm}}$. This result is an estimate: it is calculated using a rounded length.

Check: $\frac{6}{8} = \frac{3,45}{4,6} = 0,75$: the ratios are equal. The scale factor is strictly less than 1, so each side of PNM must be strictly shorter than its corresponding side in JKL : this is the case, $6 < 8$ and $3,45 < 4,6$. ✓

6.5. Brevet 5: The flat floor plan

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Method. Associated model: **A2** (questions a) and b)).

 a) From the plan to the real flat, lengths are multiplied by 50.

I am looking for a length: it is multiplied by 50.

$12 \times 50 = 600$ cm, and $600 \text{ cm} = 6 \text{ m}$.

Conclusion: the living room is 6 m long.

Check: $600 \div 50 = 12$. ✓

 b) From the plan to the real flat, lengths are multiplied by 50.

I am looking for an area: it is multiplied by $50^2 = 2500$.

$48 \times 2500 = 120\,000 \text{ cm}^2$, and $120\,000 \text{ cm}^2 = 12 \text{ m}^2$, since $1 \text{ m}^2 = 10\,000 \text{ cm}^2$.

Conclusion: the bedroom has a real area of 12 m².

Check: $120\,000 \div 2500 = 48$. ✓

 c) The piece of data “the ceiling height is 2,50 m” was not used: a floor plan is a plane figure representing the floor, so heights do not appear on it. A problem statement may contain an unneeded piece of data: one must choose those that are useful.

6.6. Brevet 6: The 1/18 scale model car

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Method. Associated model: **A2**.

 a) From the car to the scale model, lengths are multiplied by $\frac{1}{18}$, that is, divided by 18.

I am looking for a length: it is divided by 18.

$4,32 \div 18 = 0,24$ m, and $0,24 \text{ m} = 24 \text{ cm}$.

Conclusion: the scale model is 24 cm long.

Check: $0,24 \times 18 = 4,32$. ✓

 b) From the car to the scale model, lengths are divided by 18.

I am looking for an area: it is divided by $18^2 = 324$.

$1,296 \div 324 = 0,004 \text{ m}^2$, and $0,004 \text{ m}^2 = 40 \text{ cm}^2$, since $1 \text{ m}^2 = 10\,000 \text{ cm}^2$.

Conclusion: the windscreen of the scale model has an area of 40 cm².

Check: $0,004 \times 324 = 1,296$. ✓

 c) From the car to the scale model, lengths are divided by 18.

I am looking for a volume: it is divided by $18^3 = 5\,832$.

$291,6 \div 5\,832 = 0,05 \text{ L}$, and $0,05 \text{ L} = 50 \text{ mL}$, since $1 \text{ L} = 1\,000 \text{ mL}$.

Conclusion: the boot of the scale model has a volume of 50 mL.

Check: $0,05 \times 5\,832 = 291,6$. ✓

Warning! In b), dividing by 18 instead of 18^2 would give $1,296 \div 18 = 0,072 \text{ m}^2$, that is 720 cm^2 : a windscreen larger than an A4 sheet for a 24 cm-long scale model is implausible.

6.7. Brevet 7: The two trays

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a) The perimeter is a sum of lengths: if each side is multiplied by k , so is the sum. Therefore $k = \frac{P_{\text{large}}}{P_{\text{small}}} = \frac{90}{60} = \boxed{1,5}$ (enlargement).

b) In an enlargement with scale factor $k = 1,5$, areas are multiplied by $k^2 = 1,5^2 = 2,25$: the area of the large tray is $150 \times 2,25 = \boxed{337,5 \text{ cm}^2}$.

c) If the price follows the area (the amount of wood), the large one should cost $24 \times 2,25 = \boxed{54 \text{ €}}$, and not $24 \times 1,5 = 36 \text{ €}$. The craftsman who sells at “ $\times 1,5$ ” penalises himself: he gives 2,25 times as much wood for 1,5 times as much money.

6.8. Brevet 8: Photo formats

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- a) $\frac{40}{10} = 4$ and $\frac{60}{15} = 4$: both dimensions are multiplied by the same number, so yes, it is an enlargement with scale factor $\boxed{4}$.
- b) In an enlargement with scale factor 4, areas are multiplied by $4^2 = \boxed{16}$ (the print goes from 150 cm^2 to 2400 cm^2).
- c) $\frac{13}{10} = 1,3$ but $\frac{18}{15} = 1,2$: the ratios differ, so the 13×18 is not an enlargement of the 10×15 (the proportions change). To fill the 13×18 paper without distorting, the lab first enlarges by 1,3 (obtaining $13 \times 19,5$), then crops 1,5 cm off the length: that is why edges of the image disappear.

6.8. Optional extension

Detailed solution: recovering the same format after cutting

A1. Half the area does not enforce the same shape

The 10×15 sheet has an aspect ratio of $15/10 = 3/2$. One half measures $10 \times 7,5$: its long side is 10 and its short side is 7,5. Its ratio is therefore $10/7,5 = 4/3$.

The numbers $3/2$ and $4/3$ differ: $3 \times 3 = 9 \neq 8 = 2 \times 4$. The rectangles do not have the same shape, even when rotating the half. Yet its area is $10 \times 7,5 = 75 \text{ cm}^2$, half of the initial 150 cm^2 . The rule “area multiplied by k^2 ” already assumes a reduction; one cannot recognise a reduction from the area alone.

A2. Ruling out an entire case

If $L \geq 2\ell$, then $L/2 \geq \ell$. Matching the lengths in order, the two “half / full sheet” scale factors are

$$\frac{L/2}{L} = \frac{1}{2} \quad \text{and} \quad \frac{\ell}{L} = 1.$$

They differ, so the shape changes. In the boundary case $L = 2\ell$, the half is even a square, whereas the initial sheet has one side twice the other. No sheet in this case works.

B1. Translating the new matching

For $\ell < L < 2\ell$, we have $L/2 < \ell$. The initial long side L corresponds to the long side of the half ℓ ; the initial short side ℓ corresponds to $L/2$. Preserving the shape therefore requires

$$\frac{\ell}{L} = \frac{L/2}{\ell} \implies \ell^2 = \frac{L^2}{2} \implies L^2 = 2\ell^2.$$

We multiply by 2, then divide by $\ell^2 > 0$. With $r = L/\ell$, this gives $r^2 = 2$. Since $r > 0$, the only candidate is $r = \sqrt{2}$. It is indeed between 1 and 2, since $1^2 < 2 < 2^2$. Changing the order of the sides is compatible with the solution.

B2. Checking the candidate, then calculating a sheet

Conversely, if $L^2 = 2\ell^2$ with positive lengths, we divide by $2L\ell$:

$$\frac{L}{2\ell} = \frac{\ell}{L}.$$

These are the two scale factors for corresponding lengths: they are equal, so the half indeed has the same shape. The condition is necessary and sufficient in the domain under consideration.

For $\ell = 10 \text{ cm}$, we obtain

$$\boxed{L = 10\sqrt{2} \text{ cm} \approx 14,14 \text{ cm.}}$$

The scale factor to one half is $k = \ell/L = 1/\sqrt{2} = \sqrt{2}/2$. The last equality is checked by multiplying $\sqrt{2}/2$ by $\sqrt{2}$: we find $2/2 = 1$. For the area,

$$k^2 = \frac{1}{(\sqrt{2})^2} = \frac{1}{2}.$$

This is consistent with splitting into two equal halves. A length rounded to 14,14 cm gives only an approximate realisation of the exact format $10\sqrt{2} \text{ cm}$.

B3. Repeating the cut

The first half has the same aspect ratio $\sqrt{2}$. It therefore satisfies the proven condition in turn: we can repeat the process. After two successive cuts, a piece represents $(1/2) \times (1/2) = 1/4$ of the initial area.

The length scale factors multiply: $k \times k = k^2 = 1/2$. We can also track the dimensions: after the second cut, a piece measures $L/2$ by $\ell/2$. Each side has been divided by 2, and the area by 4.

C1. Applying the same approach with three pieces

In the domain $\ell < L < 3\ell$, the short side of the piece is $L/3$ and its long side is ℓ . We therefore need

$$\frac{\ell}{L} = \frac{L/3}{\ell} \implies L^2 = 3\ell^2 \implies r^2 = 3.$$

The only positive candidate is $r = \sqrt{3}$. It belongs to the domain: $1 < \sqrt{3} < 3$. Conversely, $L^2 = 3\ell^2$ indeed gives the equality of ratios on dividing by $3L\ell > 0$. This format works:

$$\boxed{r = \sqrt{3}, \quad k = \frac{1}{\sqrt{3}}, \quad \frac{\text{area of one piece}}{\text{initial area}} = \frac{1}{3}.$$

The method transfers because we redo the matching and the proof; we do not keep the $\sqrt{2}$ format from the first cut.

C2. Rounding can mask an exact discrepancy

With the exact measurements given,

$29,7^2 = 882,09 \quad \text{and} \quad 2 \times 21^2 = 2 \times 441 = 882.$

These numbers are not equal. The condition $L^2 = 2\ell^2$ is therefore not satisfied: the sheet does not preserve its format exactly. The two formats are very close, which explains why a measurement or a drawing may not show the gap. “Almost equal” and “equal” remain two different conclusions.

6.9. Brevet 9: The hidden common angle

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a) We know that $\widehat{EAD} = \widehat{CAB}$ (common angle at A , with $E \in [AB]$ and $D \in [AC]$) and that $\widehat{AED} = \widehat{ACB}$ (given in the problem statement). Now, if two triangles have two pairs of equal angles, then they are similar: triangles AED and ACB are similar, with $A \leftrightarrow A$, $E \leftrightarrow C$, $D \leftrightarrow B$. ■

b) Corresponding sides: $[AE] \leftrightarrow [AC]$, $[ED] \leftrightarrow [CB]$, $[AD] \leftrightarrow [AB]$. The reduction scale factor from ACB to AED is $k = \frac{AE}{AC} = \frac{3}{9} = \frac{1}{3}$. Therefore:

$$ED = \frac{1}{3} \times CB = \frac{1}{3} \times 7,5 = \boxed{2,5} \quad AB = AD \div \frac{1}{3} = 4 \times 3 = \boxed{12}$$

(for AB , we work back from the reduced triangle to the initial triangle: we divide by $\frac{1}{3}$, that is, we multiply by 3).

c) $AE = 3$ and $AB = 12$: we indeed have $3 < 12$, so E lies inside $[AB]$ ✓. This check is useful because reversing the direction of calculation (multiplying AD by $\frac{1}{3}$ instead of dividing, which would give $AB = \frac{4}{3} < 3 = AE$) would yield a length incompatible with $E \in [AB]$.

7. Going further (sorted bank)

7.1. Geography: reading a map

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Method. Associated model: **A2** (questions a) and c)).

a) From the map to reality, lengths are multiplied by 2 000 000.
I am looking for a length: it is multiplied by 2 000 000.
 $19,6 \times 2\,000\,000 = 39\,200\,000$ cm, and $39\,200\,000$ cm = 392 km, because 1 km = 100 000 cm.
Conclusion: as the crow flies, Paris and Lyon are 392 km apart.
Check: $39\,200\,000 \div 2\,000\,000 = 19,6$. ✓

b) The map gives the distance as the crow flies, in a straight line. The road goes around terrain and towns: it is longer, 465 km > 392 km. The two answers do not measure the same thing.

c) From the map to reality, lengths are multiplied by 200 000.
I am looking for an area: it is multiplied by $200\,000^2 = (2 \times 10^5)^2 = 4 \times 10^{10}$.
 $4,5 \times 4 \times 10^{10} = 18 \times 10^{10} = 1,8 \times 10^{11}$ cm², and $1,8 \times 10^{11}$ cm² = 18 km², because 1 km² = 10¹⁰ cm² and $1,8 \times 10^{11} \div 10^{10} = 1,8 \times 10 = 18$.
Conclusion: the lake has an actual area of 18 km².
Check: $1,8 \times 10^{11} \div (4 \times 10^{10}) = 4,5$. ✓

7.2. Biology: under the microscope

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Method. Associated model: **A2** (questions a) and b)).

a) From the cell to its image, lengths are multiplied by 400. I am looking for a length: it is multiplied by 400. $50 \times 400 = 20\,000$ μm, and $20\,000$ μm = 20 mm = 2 cm, because 1 μm = 0,001 mm and 1 cm = 10 mm. <u>Conclusion:</u> on the printed photograph, the cell measures 2 cm. <u>Check:</u> $20\,000 \div 400 = 50$. ✓	b) Lengths are multiplied by 400. I am looking for an area: it is multiplied by $400^2 = 160\,000$. <u>Conclusion:</u> the area of the image of the cell is multiplied by 160 000.
--	---

c) $1\text{ }\mu\text{m} = 0,001\text{ mm} = 10^{-3}\text{ mm}$ and $1\text{ mm} = 10^{-3}\text{ m}$, so $1\text{ }\mu\text{m} = 10^{-3} \times 10^{-3}\text{ m} = 10^{-6}\text{ m}$.
 $50\text{ }\mu\text{m} = 50 \times 10^{-6}\text{ m} = 5 \times 10^1 \times 10^{-6}\text{ m} = \boxed{5 \times 10^{-5}\text{ m}}$, because $10^1 \times 10^{-6} = 10^{1+(-6)} = 10^{-5}$.

7.3. The impossible giant 🏰

★★★★

- a) Height is a length, multiplied by 10: $1,80 \times 10 = \boxed{18 \text{ m}}$.
- b) Mass follows volume, multiplied by $10^3 = 1\,000$: $80 \times 1\,000 = 80\,000 \text{ kg} = \boxed{80 \text{ tonnes}}$.
- c) The cross-sectional area is multiplied by $10^2 = \boxed{100}$.
- d) The mass is 1 000 times greater, the strength only 100 times: each cm^2 of bone carries $\frac{1\,000}{100} = \boxed{10}$ times more mass. The load per unit of cross-section would therefore be ten times greater: the giant would be more fragile. However, one cannot state that the bones would break without knowing the maximum load they initially supported. This is the “square-cube law”: it also explains why an elephant has column-like legs and a gazelle has slender legs (the larger an animal is, the thicker its bones must be in proportion).

7.4. Transformations to get from one to the other 🏰

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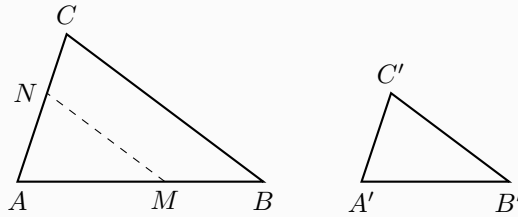
- a) On the grid, the legs of the right angle measure 2 and 1 for T_1 , 4 and 2 for T_2 : they are doubled. For the hypotenuses, Pythagoras’ theorem gives $BC^2 = 2^2 + 1^2 = 5$ and $B'C'^2 = 4^2 + 2^2 = 20 = 4 \times 5$: the square is multiplied by 4, so the hypotenuse is multiplied by 2 (since $(2\ell)^2 = 4\ell^2$). All three sides are doubled: T_2 is similar to T_1 , with scale factor $\boxed{k = 2}$ (enlargement).
- b) Rotate T_1 by a quarter turn anticlockwise about A (the sides of lengths 2 and 1 become vertical and horizontal, as in T_2), then slide it to bring A onto A' , then enlarge it to double the size with centre A' .

To remember. The “enlargement with a fixed centre” mentioned in the problem is a homothety: in this example, a rotation, a translation and an enlargement are enough (for two mirror-image triangles, a reflection across a line may also be needed).

7.5. The converse of the characterisation 🏰

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- a) Place M on $[AB]$ with $AM = A'B'$; this point exists since $0 < A'B' < AB$. Then draw the line parallel to (BC) passing through M , which intersects $[AC]$ at N .



- b) $M \in [AB]$, $N \in [AC]$ and $(MN) \parallel (BC)$: by proportionality in nested triangles (the intercept theorem (Thales’ theorem in France), seen in Year 9), $\frac{AM}{AB} = \frac{AN}{AC} = \frac{MN}{BC}$.
- c) By construction $AM = A'B'$, so $\frac{AM}{AB} = \frac{A'B'}{AB} = k$. The three ratios in question b) are therefore equal to k . Thus $AN = k \times AC$; yet the hypothesis also gives $A'C' = k \times AC$; hence $AN = A'C'$. Similarly $MN = k \times BC = B'C'$.
- d) Triangles AMN and $A'B'C'$ have all three pairs of sides equal in length ($AM = A'B'$, $AN = A'C'$, $MN = B'C'$): they are congruent (“side-side-side” case), so their angles are equal in pairs. It remains to go from AMN to ABC : the angle at A is shared by both triangles ($\widehat{MAN} = \widehat{BAC}$), and $(MN) \parallel (BC)$ gives $\widehat{AMN} = \widehat{ABC}$ and $\widehat{ANM} = \widehat{ACB}$ (corresponding angles). Therefore the angles of $A'B'C'$ have the same measures as those of AMN , which have the same measures as those of ABC : the angles of the two original triangles are equal in pairs. ■

7.6. The Sierpiński triangle 🏰

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- a) Each retained triangle has one vertex of the initial triangle and the midpoints of the two sides meeting at that vertex: each of its sides measures half the corresponding side (by definition of the midpoint and the midpoint theorem, Year 9). It is therefore a reduction with scale factor $\frac{1}{2}$, and its area is $\mathcal{A} \times \left(\frac{1}{2}\right)^2 = \boxed{\frac{\mathcal{A}}{4}}$.
- b) Step 1: $3 \times \frac{\mathcal{A}}{4} = \boxed{\frac{3}{4}\mathcal{A}}$. Step 2: each triangle undergoes the same process, so the area is multiplied by $\frac{3}{4}$: $\left(\frac{3}{4}\right)^2 \mathcal{A} = \boxed{\frac{9}{16}\mathcal{A}}$. At step n : $\boxed{\left(\frac{3}{4}\right)^n \mathcal{A}}$.
- c) Step 0: $3 \times 8 = 24 \text{ cm}$. Step 1: 3 triangles of side 4 cm, that is $3 \times 12 = \boxed{36 \text{ cm}}$. Step 2: 9 triangles of side 2 cm, that is $9 \times 6 = \boxed{54 \text{ cm}}$. The total perimeter is multiplied by $\boxed{\frac{3}{2}}$ at each step.
- d) When the operation is repeated a very large number of times, the shaded area approaches 0 (multiplied by $\frac{3}{4} < 1$ at each step), whereas the total

perimeter exceeds any value (multiplied by $\frac{3}{2} > 1$ at each step): we obtain a shape “with no area” but with a boundary of infinite length (a fractal).

7.7. Challenge: almost twins

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 **Analysis.** Yes, it is possible: the idea is to take a triangle whose sides are already in a $\times k$ progression, then multiply it by k ; the two middle lengths then overlap.

Example with $k = 1,5$: triangle T_1 with sides 4; 6; 9 and triangle T_2 with sides 6; 9; 13,5.

- They are indeed triangles: $4 + 6 = 10 > 9$ ✓ and $6 + 9 = 15 > 13,5$ ✓ (triangle inequality, essential check!).
- They are similar: $\frac{6}{4} = \frac{9}{6} = \frac{13,5}{9} = 1,5$ (enlargement).
- They are not congruent ($k = 1,5 \neq 1$), and they share two side lengths: 6 and 9. ■

Remark: it is essential to check the triangle inequality, because three numbers in geometric progression do not always form a triangle (1; 2; 4 for example, since $1 + 2 < 4$).

7.7. Optional extension

Detailed solution: from the example to an entire family

A1. Distinguishing shared length and corresponding side

Ordering the sides in the same way gives

$$\frac{6}{4} = \frac{9}{6} = \frac{13,5}{9} = \frac{3}{2}.$$

The sides of lengths 4, 6, 9 of the first correspond respectively to those of lengths 6, 9, 13,5 of the second.

Shared length	Rank in the first	Rank in the second
6	middle	shortest
9	longest	middle

A single value shifts from one rank to another. That is why two shared lengths do not force a scale factor equal to 1. However, this example does not meet the new requirement: 13,5 cm is not a whole number of centimetres.

A2. Ruling out isosceles triangles

Let a be the shortest length of the first triangle. In the enlargement, the shortest is ka . Since $a > 0$ and $k > 1$, we have $ka > a$: all lengths of the second are strictly greater than a . The value a therefore cannot be shared.

A non-equilateral isosceles triangle has only two distinct length values: if the smaller is excluded, at most one shared value remains. An equilateral triangle has only one and cannot share two distinct values either. Any solution sought therefore has three distinct side lengths: we can write them $a < b < c$.

A3. Showing why the sides follow one another by multiplication

The two ordered lists are

$$\begin{array}{c|ccc} \text{first triangle} & a & b & c \\ \text{enlargement} & ka & kb & kc \end{array} \quad (k > 1).$$

We have just proved that a does not appear in the second list. Moreover, $kc > c$, so kc is greater than all sides of the first: it cannot be shared either.

Only b, c remain in the first list, and ka, kb in the second. Since two distinct shared values are required, the two sets of values are equal: $\{b; c\} = \{ka; kb\}$. Now $b < c$ and $ka < kb$; the smaller values are therefore equal, and then the larger ones:

$$b = ka, \quad c = kb = k^2a.$$

These equalities concern the shared lengths. The corresponding sides remain paired in the same rank: a with ka , b with kb , c with kc . The two pairings must not be confused.

B1. Writing the six lengths and checking the geometry

For $k = 3/2$, the relations from A3 give:

	shortest side	middle side	longest side
first triangle	a	$\frac{3a}{2}$	$\frac{9a}{4}$
second triangle	$\frac{3a}{2}$	$\frac{9a}{4}$	$\frac{27a}{8}$

For every $a > 0$, the sum of the two shorter sides of the first is

$$a + \frac{3a}{2} = \frac{5a}{2} = \frac{10a}{4} > \frac{9a}{4}.$$

The triangle inequality is strict: the three lengths form a non-degenerate triangle. Multiplying its three sides by $3/2 > 0$ preserves this inequality: the second is also a triangle. This check is essential before interpreting the lists of numbers as geometric figures.

B2. Bringing out three compulsory factors of 2

The number a is a positive integer. For $3a/2$ to be an integer, $3a$ must be even. Since 3 is odd, a must be even: we write $a = 2u$, with u a positive integer. Next, we examine the side $9a/4$:

$$\frac{9a}{4} = \frac{9 \times 2u}{4} = \frac{9u}{2}.$$

This number can only be an integer if $9u$ is even. Since 9 is odd, u must be even: $u = 2v$. Thus $a = 4v$.

Finally, the longest side of the second triangle is

$$\frac{27a}{8} = \frac{27 \times 4v}{8} = \frac{27v}{2}.$$

For it to be an integer, $27v$ must be even. Since 27 is odd, $v = 2n$ with n a positive integer. We therefore obtain $a = 8n$. The three successive divisions by 2 explain why merely being even, or merely a multiple of 4, is not enough.

B3. Describing all solutions and checking the converse

Replacing a by $8n$, every solution must have the form

$$(8n; 12n; 18n) \quad \text{and} \quad (12n; 18n; 27n), \quad n \geq 1 \text{ an integer.}$$

This establishes a necessary condition. Let us now check that it is sufficient for every integer $n \geq 1$:

- All six lengths are strictly positive integers.
- They form two triangles: $8n + 12n = 20n > 18n$ and $12n + 18n = 30n > 27n$.
- The ratios between sides of the same rank are $12n/(8n) = 18n/(12n) = 27n/(18n) = 3/2$. The triangles are therefore similar.
- They are not congruent: their shortest sides are $8n$ and $12n$ respectively, two different values since $n > 0$.
- They share exactly the two distinct lengths $12n$ and $18n$. The value $8n$ is too small to belong to the second, and $27n$ is too large to belong to the first.

Every solution has this form, and every pair of this form works: we have indeed obtained all solutions, and not just a collection of examples.

B4. Justifying the smallest choice

The shortest side is $a = 8n$, with n an integer at least equal to 1. It is therefore at least equal to 8 cm. For $n = 1$, we obtain $(8; 12; 18)$ and $(12; 18; 27)$; B3 guarantees that this pair works. The minimum $a = 8$ cm is therefore achieved, and no smaller value is possible.

C1. Adding the 100 cm bound

The longest side of the second triangle is $27n$. We require $27n \leq 100$. Now $27 \times 3 = 81 \leq 100$ and $27 \times 4 = 108 > 100$. Since n is a positive integer, the only possibilities are 1, 2 and 3:

n	First triangle (cm)	Second triangle (cm)	Shared lengths (cm)
1	(8; 12; 18)	(12; 18; 27)	12 and 18
2	(16; 24; 36)	(24; 36; 54)	24 and 36
3	(24; 36; 54)	(36; 54; 81)	36 and 54

There are exactly three pairs. None is missing, since B3 describes all solutions and we have examined all integers allowed by the bound. They are different because their shortest sides $8n$ are different.

C2. Understanding why scale factor 2 fails

Part A applied to every $k > 1$. If two triangles with scale factor 2 shared two distinct lengths, the sides of the first would therefore be a , $2a$, $4a$, with $a > 0$. But

$$a + 2a = 3a < 4a.$$

The triangle inequality is not satisfied: these lengths cannot form a triangle. It is impossible even if non-integer lengths are allowed.

The obstacle has changed: for $3/2$, we had genuine triangles and had to select integer lengths; for 2, the construction of the triangle itself fails. The proportionality of two lists of numbers does not guarantee that those lists represent triangles.

7.8. Metric relations in a right-angled triangle: Pythagoras proved anew

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Method. Four relations, a single engine: “shared acute angle + right angle $\Rightarrow$ similar”, used twice. All the difficulty lies in the pairing: follow the angles, write the three ratios in order, then collect the cross-products.

- a) In triangles ABC and HBA , we have $\widehat{ABC} = \widehat{HBA}$ (shared angle at B , since $H \in [BC]$) and $\widehat{BAC} = \widehat{BHA} = 90^\circ$. Now, if two triangles have two pairs of equal angles, then they are similar. Therefore triangles ABC and HBA are similar. Pairing via the angles: the right angles correspond ($A \leftrightarrow H$), as does the shared angle ($B \leftrightarrow B$), so $C \leftrightarrow A$. Opposite $\widehat{BAC} = 90^\circ$: $[BC]$; opposite $\widehat{BHA} = 90^\circ$: $[BA]$. Opposite $\widehat{ABC} = \widehat{HBA}$: $[CA]$ and $[AH]$. Opposite $\widehat{ACB}$ and its corresponding angle $\widehat{BAH}$: $[AB]$ and $[HB]$. Hence

$$\frac{BC}{BA} = \frac{CA}{AH} = \frac{AB}{HB}.$$

- b) From $\frac{CA}{AH} = \frac{BC}{BA}$, cross-multiplying gives $AH \times BC = CA \times BA$, so $AH = \frac{AB \times AC}{BC}$. From $\frac{BC}{BA} = \frac{AB}{HB}$, cross-multiplying gives $AB^2 = BH \times BC$ (Euclid's theorem). ■

- c) Similarly, in triangles ABC and HAC , the angle at C is shared ($\widehat{ACB} = \widehat{HCA}$, with $H \in [BC]$) and $\widehat{BAC} = \widehat{CHA} = 90^\circ$: they are similar, with $A \leftrightarrow H$, $C \leftrightarrow C$, $B \leftrightarrow A$, hence

$$\frac{BC}{AC} = \frac{CA}{CH} = \frac{AB}{HA}.$$

From $\frac{BC}{AC} = \frac{CA}{CH}$, cross-multiplying gives $AC^2 = CH \times CB$. ■

- d) From a): $\frac{CA}{AH} = \frac{AB}{HB}$ gives $AH \times AB = CA \times HB$, that is $AH = HB \times \frac{CA}{AB}$. From c): $\frac{CA}{CH} = \frac{AB}{HA}$ gives $AH \times CA = CH \times AB$, that is $AH = CH \times \frac{AB}{CA}$. Multiplying side by side:

$$AH^2 = HB \times CH \times \frac{CA}{AB} \times \frac{AB}{CA} = HB \times HC. \quad AH^2 = HB \times HC$$

(the altitude relation, known in France as “Descartes’ relation”). ■

- e) Adding the two Euclid equalities: $AB^2 + AC^2 = BH \times BC + CH \times CB = BC \times (BH + CH)$. Since H lies on the segment $[BC]$, $BH + HC = BC$, so $AB^2 + AC^2 = BC \times BC = BC^2$. $BC^2 = AB^2 + AC^2$: this is Pythagoras’ theorem, proved anew using similar triangles alone. ■

- f) Numerical check ($AB = 3$, $AC = 4$, $BC = 5$): $AH = \frac{3 \times 4}{5} = \frac{12}{5} = 2,4$; $BH = \frac{AB^2}{BC} = \frac{9}{5} = 1,8$; $CH = \frac{AC^2}{BC} = \frac{16}{5} = 3,2$. Checks: $1,8 + 3,2 = 5 = BC$ ✓; $AH^2 = 5,76 = 1,8 \times 3,2$ ✓; $3^2 = 9 = 1,8 \times 5$ ✓; $4^2 = 16 = 3,2 \times 5$ ✓

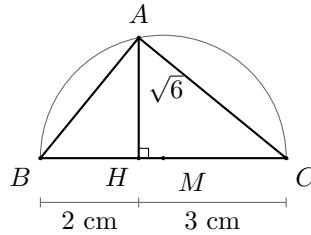
To remember. In a right-angled triangle, the altitude from the right angle creates three pairwise similar triangles (ABC , HBA , HAC): it is a goldmine of relations—altitude $AH = \frac{AB \times AC}{BC}$, Euclid $AB^2 = BH \times BC$, Descartes $AH^2 = HB \times HC$ —and a royal gateway for an oral presentation.

7.8. Optional extension

Detailed solution: constructing and proving an exact length

A1. Choosing the circle that guarantees the right angle

The diameter measures $BC = 5$ cm, so the radius is $5/2 = 2,5$ cm. Since H lies between B and C , we have $HC = BC - BH = 5 - 2 = 3$ cm. Place the centre M at 2,5 cm from B , then draw the semicircle of radius 2,5 cm. The perpendicular at H intersects the upper arc at A :



The length shown on the figure is justified below; it is not deduced by reading a ruler.

A2. Proving the length obtained

Point A lies on the circle with diameter $[BC]$ and is distinct from its endpoints. The circle property therefore gives $\widehat{BAC} = 90^\circ$: ABC is right-angled at A .

By construction, $H \in [BC]$ and $(AH) \perp (BC)$: $[AH]$ is the altitude from the right angle. The hypotheses of the relation proved in 7.8 are met:

$$AH^2 = BH \times HC = 2 \times 3 = 6 \text{ cm}^2.$$

Since $AH > 0$, we obtain $AH = \sqrt{6} \text{ cm}$. Finally, $2^2 = 4 < 6 < 9 = 3^2$, so $2 < \sqrt{6} < 3$. This bound confirms the order of magnitude on the figure, without replacing the proof.

A3. Distinguishing the construction from its measurement

We calculate $2,45^2 = 6,0025 \neq 6$. The number 2,45 therefore cannot be the exact value of AH ; it is an approximation of it.

The proof is the chain “point on the circle with diameter $[BC]$; right-angled triangle; altitude from the right angle; relation $AH^2 = BH \times HC$ ”. It is the properties guaranteed by the construction that determine the exact length. A ruler graduation only gives an approximate check.

B1. Making the construction work for two positive lengths

With $BH = p$ cm and $HC = q$ cm, we draw a diameter of length $(p + q)$ cm and a semicircle of radius $(p + q)/2$ cm. Since p and q are strictly positive, H lies strictly between B and C ; the perpendicular at H indeed meets the upper arc at a point A .

Triangle ABC is right-angled at A , and $[AH]$ is its altitude to the hypotenuse. If h denotes the length of AH in centimetres, the same relation gives

$$h^2 = pq \quad \text{and therefore} \quad AH = \sqrt{pq} \text{ cm.}$$

We have applied the hypotheses of the theorem to arbitrary p, q , rather than extrapolating from the single trial (2; 3). The product governs the altitude; the sum governs the diameter.

B2. The same altitude does not force similar triangles

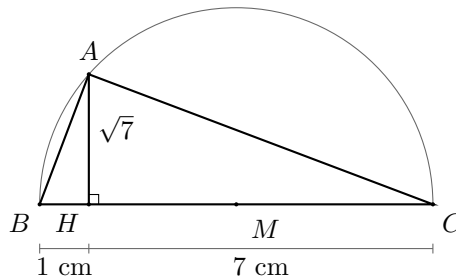
For $p = 1$ and $q = 6$, the diameter is 7 cm, the radius 3,5 cm and $h^2 = 1 \times 6 = 6$. We therefore again construct an altitude of length $\sqrt{6}$ cm.

Suppose the two large triangles were similar. Since they are right-angled, their hypotenuses, which are the longest sides, would correspond. The scale factor from the triangle of diameter 5 to that of diameter 7 would therefore be $k = 7/5$. The altitude from the right angle would also be multiplied by $7/5$; it would measure $(7/5)\sqrt{6}$ cm. Yet it still measures $\sqrt{6}$ cm and $\sqrt{6} > 0$: this is incompatible with $7/5 \neq 1$.

The triangles are therefore not similar. Two different constructions can produce the same useful length without preserving all the proportions of their figures.

C1. Designing a construction of $\sqrt{7}$

We want $pq = 7$. A simple choice is $p = 1$ and $q = 7$. We therefore draw $BC = 8$ cm, place H at 1 cm from B , then draw the semicircle with centre M , the midpoint of $[BC]$, and radius 4 cm. The perpendicular at H meets the arc at A :



The conditions of B1 are satisfied: $1 > 0$, $7 > 0$, H interior to the diameter. Thus $AH^2 = 1 \times 7 = 7 \text{ cm}^2$, so $AH = \sqrt{7}$ cm. Another positive choice with product 7 would also work; the problem did not ask to describe them all.

C2. Proving a maximum and showing that it is achieved

The diameter measures 10 cm: the radius AM is always 5 cm. If $H = M$, the altitude is precisely the vertical radius; it measures $AH = 5$ cm. This choice is allowed: $BH = HC = 5$ cm are strictly positive.

If $H \neq M$, points H, M lie on (BC) and $(AH) \perp (BC)$. Triangle AHM is therefore right-angled at H . Pythagoras gives

$$AH^2 + HM^2 = AM^2 = 25 \text{ cm}^2.$$

Since $HM > 0$, we have $HM^2 > 0$, so $AH^2 < 25 \text{ cm}^2$ and $AH < 5$ cm. No other position can reach or exceed 5 cm.

$$AH_{\max} = 5 \text{ cm,} \quad \text{only for } H = M.$$

We have proved the two points required for a maximum: 5 cm is achievable, and every other altitude is strictly smaller. With a fixed sum $p + q = 10$, the construction thus makes visible why the product pq is maximal for $p = q = 5$.

7.9. Challenge: Pythagoras via the circle

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- a) $[BE]$ and $[BC]$ are two radii of the circle with centre B passing through C : $BE = BC = a$, and similarly $BD = a$. We are sure that $c < a$ because, in triangle ABC right-angled at A , $[BC]$ is the hypotenuse, hence the longest side. Thus A (at distance c from B) lies inside the circle, and:

$$AE = BE - BA = a - c \quad \text{and} \quad AD = DB + BA = a + c.$$

- b) Points D and E lie on the circle and are collinear with its centre B : $[DE]$ is a diameter. Point C also lies on the circle. Now, if a triangle is inscribed in a circle with one side as a diameter, then it is right-angled. Therefore triangle DEC is right-angled at C (the right angle faces the diameter). ■

- c) Triangles CAE and DAC are both right-angled at A : $(CA) \perp (AB)$, and points D, B, A, E are collinear on (AB) . In triangle DEC right-angled at C , we have $\widehat{CDE} + \widehat{CED} = 90^\circ$. In triangle DAC right-angled at A , we have $\widehat{CDA} + \widehat{DCA} = 90^\circ$. Since $\widehat{CDE} = \widehat{CDA}$ (it is the same angle: A lies on segment $[DE]$), we deduce $\widehat{CED} = \widehat{DCA}$. Now $A \in [DE]$, so $\widehat{AEC} = \widehat{CED} = \widehat{DCA}$. Triangles CAE and DAC therefore have two pairs of equal angles (90° at A , and $\widehat{AEC} = \widehat{DCA}$): they are similar, with $C \leftrightarrow D$, $A \leftrightarrow A$, $E \leftrightarrow C$. Hence, writing the ratios of the corresponding sides:

$$\frac{AE}{AC} = \frac{CA}{DA}, \quad \text{that is,} \quad \frac{a - c}{b} = \frac{b}{a + c}.$$

Cross-multiplying gives $(a - c)(a + c) = b^2$. Expanding $(a - c)(a + c) = a^2 + ac - ca - c^2 = a^2 - c^2$, we obtain $a^2 - c^2 = b^2$, so

$$a^2 = b^2 + c^2 \quad \blacksquare$$

To remember. This proof connects three tools: similar triangles, a triangle inscribed in a semicircle and the identity $(a - c)(a + c) = a^2 - c^2$.

7.10. Two proofs of Pythagoras: which one sheds light?

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- The four sides of the central region are the hypotenuses of the four triangles: they all measure c . At each vertex, the straight angle (180°) is shared between the two acute angles of two identical right-angled triangles (one of each measure, hence complementary, adding up to 90°) and the angle of the central region, which is therefore $180^\circ - 90^\circ = 90^\circ$. Four sides of equal length and four right angles: it is a square of side c , with area c^2 .
- On the right, the four triangles are paired along their hypotenuses to form two rectangles with sides a and b . Inside the large square of side $a + b$, the two remaining white regions have four right angles and sides of length $(a + b) - b = a$ for one and $(a + b) - a = b$ for the other: they are a square of side a (area a^2) and a square of side b (area b^2).
- Both figures start from the same large square of side $a + b$ and contain the same four triangles: the remaining white area is therefore the same on both sides. Hence

$$c^2 = a^2 + b^2. \quad \blacksquare$$

We can also read it as an area balance: $(a + b)^2 = c^2 + 4 \times \frac{ab}{2}$, that is $a^2 + 2ab + b^2 = c^2 + 2ab$, hence $a^2 + b^2 = c^2$.

- By cutting out and moving the four triangles, one checks “by hand” that the remaining white area does not change.
- Both proofs are valid. The puzzle proof convinces quickly and shows why (we watch the same area rearrange itself); the proof using similar triangles (exercise 7.8) also convinces and, as a bonus, generalises: along the way it provides Euclid’s relations and the altitude relation. “Elegant” is not “correct”: it is an additional judgement.

To remember. Both proofs are correct; their value may differ. Dissection makes an equality of areas visible; similarity also provides relations on the altitude and the sides.

End of worked solutions. The  boxes offer justified write-ups. The H exercises and extensions remain optional.