

Chapter 01A: Similar triangles

Enlargements and reductions



Starting point

In Year 9, you learnt to recognise congruent triangles: same lengths, same angles, two triangles that can be superimposed. This chapter takes the next step: comparing triangles that have the same shape without having the same size. A photograph and its enlargement, a map and the actual ground, a scale model and the real aircraft: everywhere, the same question arises—what is preserved, and what is multiplied?

- We gain a powerful proof tool: two pairs of equal angles are enough to guarantee that two triangles have the same shape, and therefore that their sides are proportional.
- We gain a calculation tool: the inaccessible lengths of a triangle can be deduced from those of a measurable triangle.
- We gain a common-sense reflex: in an enlargement or a reduction with scale factor k , lengths are multiplied by k , whereas areas are multiplied by k^2 and volumes by k^3 .

This chapter is a cornerstone of the year: the “similar triangles” problem is a classic of Year 10, the intercept theorem (Thales’ theorem in France, chapter 01C) will appear as a special case of what we prove here, and trigonometry (chapter 08) rests on this idea of preserved shape.

Fluency skills

To be done mentally, without written working, then repeated from memory; the upside-down answers below are for self-checking. Questions 1 to 4 review previous chapters.

1. Simplify $\frac{3,5}{2,1}$.

2. Calculate $\frac{2}{3} \times \frac{9}{4}$.

3. Calculate $\left(\frac{5}{6}\right)^2$ and $\left(\frac{1}{10}\right)^3$.

4. Write 225 as a product of prime factors.

5. Two angles of a triangle measure 54° and 81° . What is the third?

6. A right-angled triangle has an angle of 25° . What are its other two angles?

7. State the three congruence cases for triangles (Year 9).

8. Calculate 8^2+15^2 and 17^2 . What can be said about a triangle with sides 8, 15, 17?

9. On a plan at a scale of $\frac{1}{250}$, what does 1 cm represent?

10. A 3×5 rectangle: both dimensions are doubled. What is the new area?

Answers (for self-checking). 1) $\frac{5}{3}$ 2) $\frac{3}{2}$ 3) $\frac{25}{36}$ and $\frac{1}{1000}$ 4) $3^2 \times 5^2$ 5) 45° 6) 90° and 65° 7) three sides; an angle included between two sides; or 4 times the initial area. 8) $64 + 225 = 289 = 17^2$: right-angled (converse of Pythagoras' theorem) 9) 250 cm, or 2,5 m 10) $6 \times 10 = 60$, a side included between two angles

1 Congruent triangles, angles and parallel lines: review

Before comparing triangles of different sizes, let us review the strictest comparison: congruence.

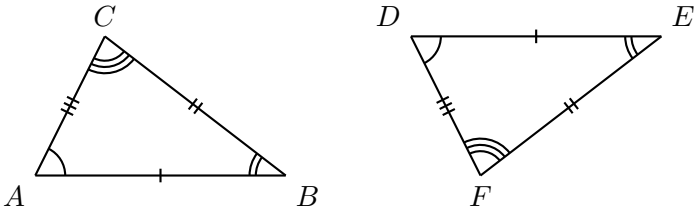
1.1 Congruent triangles

Congruent triangles

Two triangles are congruent (also called superimposable or isometric) when they can be made to coincide exactly, if necessary by sliding, rotating or flipping one of them.

Two congruent triangles have pairs of sides of equal length and pairs of angles of equal measure. The vertices that coincide are called corresponding.

In plain words: two congruent triangles are two copies of the same triangle, possibly moved or flipped; nothing distinguishes them once one is placed on top of the other.



Two congruent triangles in different positions: $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$.

In Year 8 (chapter Constructing triangles), you saw that it is not necessary to know all six measurements of a triangle, namely the lengths of its three sides and the measures of its three angles, in order to construct it: three well-chosen pieces of information are enough. That is precisely what the congruence cases, studied in Year 9, state:

The three congruence cases for triangles (Year 9)

Two triangles are congruent as soon as one of the following three conditions holds:

Case 1 “side-side-side”	Case 2 “side-angle-side”	Case 3 “angle-side-angle”
their three sides are equal in length in pairs	an equal angle included between two pairs of sides of equal length	a side of equal length included between two pairs of equal angles

▲ Warning

In case 2, the angle must be included between the two sides. With a non-included angle (two sides and an “awkwardly placed” angle), it is sometimes possible to construct two different triangles: the information is not enough. This is the counterexample to keep in mind.

Why do we need this here?

The congruence cases are used to prove the angle characterisation in Part 2. We use the “angle-side-angle” case: to compare two triangles of different sizes, we construct an intermediate triangle congruent to the smaller one, placed inside the larger one.

1.2 Angles and parallel lines (Year 8)

Vertically opposite angles, corresponding angles, alternate interior angles

Two vertically opposite angles have the same vertex and their sides are extensions of each other, in pairs. They share no common side. In the figure: $\widehat{VAM}$ and $\widehat{UAB}$.

A line is a transversal to two lines if it intersects them at two distinct points, one point on each of the two lines. Here, (MN) intersects (UV) at A and (RS) at B .

Two angles are corresponding when they:

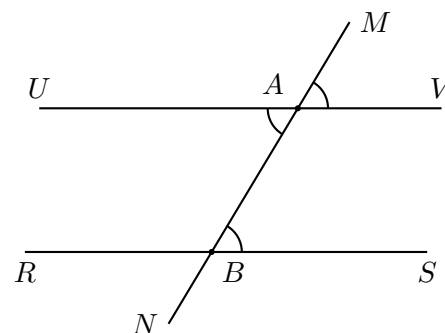
- have different vertices, A and B ;
- lie on the same side of the transversal;
- have one between the two lines and the other outside.

In the figure: $\widehat{VAM}$ and $\widehat{SBA}$.

Two angles are alternate interior when they:

- have different vertices, A and B ;
- lie on opposite sides of the transversal (“alternate”);
- lie between the two lines (“interior”).

In the figure: $\widehat{UAB}$ and $\widehat{SBA}$.



Here, $(UV) \parallel (RS)$; the transversal is (MN) .

These definitions do not require the two lines to be parallel: the position of the angles makes it possible to name them, but is not enough to state that their measures are equal.

For $(UV) \parallel (RS)$, it reads: “the line UV is parallel to the line RS ”. In plain words: these two lines do not intersect; they have the same direction, like two lines in a notebook.

Angles of equal measure

- Two vertically opposite angles have the same measure.
- If two parallel lines are cut by a transversal, then the corresponding angles they form have the same measure, and so do the alternate interior angles.

Why is this true?

The points U, A, V are collinear in this order, as are M, A, B : $\widehat{VAM} + \widehat{MAU} = 180^\circ$ and $\widehat{MAU} + \widehat{UAB} = 180^\circ$. Therefore $\widehat{VAM} = 180^\circ - \widehat{MAU} = \widehat{UAB}$: vertically opposite angles have the same measure. For parallel lines, the idea is as follows: slide the line (UV) along the transversal to B : since it is parallel to (RS) , it keeps its direction and lands on (RS) , and the angle $\widehat{VAM}$ lands exactly on $\widehat{SBA}$. Finally, $\widehat{UAB} = \widehat{VAM}$ (vertically opposite) and $\widehat{VAM} = \widehat{SBA}$ (corresponding), so $\widehat{UAB} = \widehat{SBA}$. ■

1.3 Writing a proof

Proof

A proof starts from known facts, the hypotheses (given information in the problem, results already proved), and deduces a conclusion, without assuming anything else. Each step is written in three parts:

- “We know that ...”: the useful hypotheses;
- “Now, ...”: the property, definition or rule that applies, stated in full;
- “Therefore ...”: what it gives on the figure.

The conclusion of one step can serve as a hypothesis for the next step.

For ■, it reads: “end of the proof”. In plain words: the black square, placed after the last sentence, indicates that the stated result has been established.

Writing model J1: proving a property (we know that, now, therefore)

The steps ① We know that: write down the useful hypotheses, taken from the problem or from a previous question. ② Now: state in full the property, definition or calculation rule that applies. ③ Therefore: write down what it gives on the figure. ④ Chain steps if needed: the conclusion obtained becomes a given fact; repeat until the required result is reached, then finish with ■. Here, a single step leads to the result: the written solution ends with ■ at step ③. Once the triangles are congruent, we can state without measuring that $BC = EF$, $AC = DF$ and $\widehat{ACB} = \widehat{DFE}$.	On your paper Triangles ABC and DEF satisfy $AB = DE = 5$ cm, $\widehat{BAC} = \widehat{EDF} = 40^\circ$ and $\widehat{ABC} = \widehat{DEF} = 75^\circ$. Prove that these triangles are congruent. ① We know that $\widehat{BAC} = \widehat{EDF} = 40^\circ$, that $AB = DE = 5$ cm and that $\widehat{ABC} = \widehat{DEF} = 75^\circ$. The side $[AB]$ is included between the angles $\widehat{BAC}$ and $\widehat{ABC}$, and the side $[DE]$ between the angles $\widehat{EDF}$ and $\widehat{DEF}$. ② Now, if two triangles have a side of equal length included between two pairs of equal angles, then they are congruent (“angle-side-angle” case). ③ Therefore triangles ABC and DEF are congruent. ■
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→ Exercise sheet: exercise 5.4 a) 🧑‍🎓 optional (model J1)

Writing model D1: proving that two angles have the same measure (parallel lines)

The steps

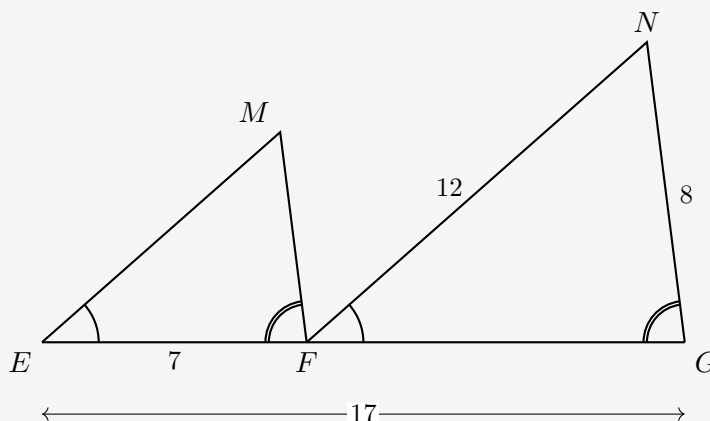
- ① We know that: state the two parallel lines and the transversal that cuts them, with the two points of intersection; state whether the two angles are corresponding or alternate interior.
- ② Now: state the property: if two parallel lines are cut by a transversal, then the corresponding (or alternate interior) angles they form have the same measure.
- ③ Therefore: write the equality of their measures.

With alternate interior angles, the wording is the same: replace the word “corresponding” with “alternate interior”.

This example serves as a running thread: question 2 is written using model D2 (section 2.2), question 3 using model L1 (section 3.1).

On your paper

The points E, F, G are collinear in this order; triangles EFM and FGN lie on the same side of the line (EG) ; $(EM) \parallel (FN)$ and $(FM) \parallel (GN)$; $EF = 7$, $EG = 17$, $FN = 12$ and $GN = 8$.



1. Prove that the angles $\widehat{FEM}$ and $\widehat{GFN}$ have the same measure.
2. Prove that triangles EFM and FGN are similar.
3. Deduce the lengths EM and FM .

1. ① We know that the lines (EM) and (FN) are parallel, and that the line (EG) intersects them at E and at F ; the angles $\widehat{FEM}$ and $\widehat{GFN}$ are corresponding.
- ② Now, if two parallel lines are cut by a transversal, then the corresponding angles they form have the same measure.
- ③ Therefore $\widehat{FEM} = \widehat{GFN}$. ■

→ Exercise sheet: exercises 2.2 a) and 6.1 a) (model D1)

1.4 Isosceles triangle, nested triangles, right-angled triangle

The angles of an isosceles triangle

An isosceles triangle has two sides of equal length; their common vertex is the apex (or main vertex), and the third side is the base. The two angles whose vertices lie on the base are the base angles; the third is the apex angle.

- If a triangle is isosceles, then its two base angles have the same measure.
- Conversely, if a triangle has two angles of equal measure, then it is isosceles, and its two sides of equal length meet at the third vertex.

In plain words: an isosceles triangle is symmetrical about the perpendicular bisector of its base, and this symmetry swaps the two base angles.

Proportionality in nested triangles (the intercept theorem (Thales' theorem in France), Year 9)

In a triangle ABC , if $M \in [AB]$, $N \in [AC]$ and $(MN) \parallel (BC)$, then $\frac{AM}{AB} = \frac{AN}{AC} = \frac{MN}{BC}$.

This theorem will be revisited and extended in chapter 01C; in this chapter, it is used only at the heart of the proofs in Part 2.

Converse of Pythagoras' theorem

In a right-angled triangle, the side opposite the right angle is the hypotenuse; it is the longest side.

If, in a triangle, the square of the longest side is equal to the sum of the squares of the other two sides, then this triangle is right-angled, and the right angle is opposite the longest side.

For BC^2 , it reads: “ BC squared”. In plain words: it is $BC \times BC$, the square of the length BC ; if $BC = 13$, then $BC^2 = 169$.

Why is this true?

Let ABC be a triangle whose longest side is $[BC]$, with $BC^2 = AB^2 + AC^2$. Construct a triangle $A'B'C'$ right-angled at A' , with $A'B' = AB$ and $A'C' = AC$. By Pythagoras' theorem, $B'C'^2 = A'B'^2 + A'C'^2 = AB^2 + AC^2 = BC^2$, so $B'C' = BC$. Triangles ABC

and $A'B'C'$ have their three sides equal in length in pairs: they are congruent (“side-side-side” case), and the angle $\widehat{BAC}$ is a right angle like $\widehat{B'A'C'}$. ■

In a written solution, we calculate the two sides of the equation separately: we do not yet know whether they are equal, and writing the equality from the outset would mean assuming what we want to prove.

Writing model J2: proving that a triangle is right-angled (converse of Pythagoras’ theorem)

The steps ① Identify the longest side. ② Calculate separately the square of the longest side and the sum of the squares of the other two sides. ③ Compare; if they are equal, conclude using the converse of Pythagoras’ theorem: the triangle is right-angled, and the right angle is opposite the longest side.	On your paper Triangle PQR satisfies $PQ = 20$, $QR = 21$ and $PR = 29$. Prove that this triangle is right-angled. ① In triangle PQR, the longest side is $[PR]$. ② On the one hand, $PR^2 = 29^2 = 841$. On the other hand, $PQ^2 + QR^2 = 20^2 + 21^2 = 400 + 441 = 841$. ③ Thus $PR^2 = PQ^2 + QR^2$. Now, if the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then this triangle is right-angled, and the right angle is opposite the longest side (converse of Pythagoras’ theorem). Therefore triangle PQR is right-angled at Q. ■
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→ Exercise sheet: exercises 2.4 a) and 4.1 d) (model J2)

Quick questions (unaided)

1. Two triangles have their three angles equal in measure in pairs. Are they necessarily congruent?

2. ABC and RST satisfy $AB = RS$, $AC = RT$ and $\widehat{BAC} = \widehat{SRT}$. Are they congruent? State the case used.

3. In triangles ABC and DEF , we know that $AB = DE$ and $\widehat{A} = \widehat{D}$. What piece of information is missing to conclude that they are congruent?

Answers: 1) no, same shape, but possibly different sizes (that is the whole point of Part 2!); 2) yes, “side-angle-side” case (angle $\widehat{A}$ is included between $[AB]$ and $[AC]$); 3) a third piece of information is missing, for example $\widehat{B} = \widehat{E}$ (“angle-side-angle” case) or $\widehat{C} = \widehat{F}$ (“side-angle-side” case).

2 Similar triangles

Introductory activity: same shape, not the same size

Here are two triangles whose angles are equal in measure in pairs: 65° , 45° and 70° .

1. Measure the six sides with a ruler, to the nearest millimetre, then copy and complete the table:

	side opposite 70°	side opposite 65°	side opposite 45°
triangle ABC	$AB = \dots$	$BC = \dots$	$AC = \dots$
triangle $A'B'C'$	$A'B' = \dots$	$B'C' = \dots$	$A'C' = \dots$
quotient	$\frac{A'B'}{AB} = \dots$	$\frac{B'C'}{BC} = \dots$	$\frac{A'C'}{AC} = \dots$

2. A conjecture is a statement thought to be true based on observations or measurements, but not yet proved. What do you notice about the three quotients? State a conjecture starting with: “If two triangles have pairs of equal angles, then...”.
3. Follow-up question: why did we compare the sides opposite equal angles, rather than pairing the sides at random?
- The conjecture will be stated, and then proved, in the rest of this section: three measurements prove nothing!

2.1 Definition and vocabulary

Similar triangles

Two triangles are similar when their vertices can be matched in pairs so that the lengths of the corresponding sides are proportional. The constant of proportionality k that takes the sides of the first triangle to the sides of the second is called the scale factor (or similarity ratio). It is a strictly positive number: $k > 0$.

The vertices that match are called corresponding, as are the sides that join them.

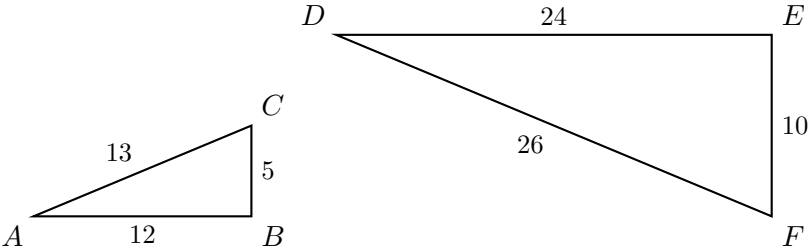
For $\frac{A'B'}{AB} = k$, it reads: “ A prime B prime over $A B$ equals k ”. In plain words: the length $A'B'$ is k times the length AB , and the same holds for each of the other two pairs of corresponding sides. The number k depends on the chosen direction: from ABC to $A'B'C'$, it is $\frac{A'B'}{AB}$; in the other direction, it is $\frac{AB}{A'B'} = \frac{1}{k}$. If $k > 1$, the second triangle is an enlargement of the first; if $0 < k < 1$, a reduction; if $k = 1$, they are congruent (these terms are defined for all shapes in Part 4).

Example

A triangle ABC with sides 5 cm, 12 cm, 13 cm and a triangle DEF with sides 10 cm, 24 cm, 26 cm are similar:

$$\frac{10}{5} = \frac{24}{12} = \frac{26}{13} = 2.$$

The scale factor (from the first to the second) is $k = 2$; in the other direction, it is $\frac{1}{2}$.



Remark: in the figure, the second triangle is flipped “in a mirror”; the orientation does not matter at all, only the lengths count.

Counterexample to discuss

A triangle with sides 5, 12, 13 and a triangle with sides 7, 14, 15 are not similar, even though 2 has been added to each side:

$$\frac{7}{5} = 1,4 \qquad \frac{14}{12} \approx 1,17 \qquad \frac{15}{13} \approx 1,15.$$

The three quotients are not equal: there is no proportionality. All three ratios must be checked, not just one or two.

Remark

- Two congruent triangles are similar, with $k = 1$. Similarity is therefore a broader notion than congruence.

- Notation. When we write “triangles ABC and $A'B'C'$ are similar”, the order of the letters indicates the correspondence: $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$ (it reads: “ A corresponds to A prime”), and the corresponding sides are read in order: $[AB] \leftrightarrow [A'B']$, etc. This sentence is sometimes abbreviated as $ABC \sim A'B'C'$, which reads: “triangle ABC is similar to triangle $A'B'C'$ ”: the symbol $\sim$ never removes the need to write the vertices in the order of correspondence.
- In practice, the proportionality table of corresponding sides is the working tool:

sides of ABC	AB	AC	BC
sides of $A'B'C'$	$A'B'$	$A'C'$	$B'C'$

Why is k calculated as “final divided by initial”, in the same unit?

By definition, each side of the second triangle is k times its corresponding side in the first: $A'B' = k \times AB$. Dividing both sides by AB , which is non-zero, gives $k = \frac{A'B'}{AB}$: the final length divided by the initial corresponding length. This quotient compares two lengths; it makes sense only if they are written in the same unit. With 14 cm and 10 cm, we find $\frac{14}{10} = 1,4$; with 14 cm and 0,1 m (the same length of 10 cm), the quotient $\frac{14}{0,1} = 140$ would be wrong.

Writing model A1: calculating a scale factor k (similarity, enlargement, reduction) or a scale

The steps

- ① Choose two known corresponding lengths and write them in the same unit.
- ② Divide the final length by the initial corresponding length: this is k ; simplify.
- ③ Conclude: if $k > 1$, it is an enlargement; if $0 < k < 1$, a reduction; for a plan or a map, the scale is written $\frac{1}{n}$.

The same model is used for a scale (section 4.1).

On your paper

Triangles ABC and DEF are similar, the corresponding vertices being $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$; $AB = 10$ cm and $DE = 14$ cm. Find the scale factor from ABC to DEF .

- ① The sides $[AB]$ and $[DE]$ are corresponding, because they join the corresponding vertices $A \leftrightarrow D$ and $B \leftrightarrow E$; they are in the same unit.
- ② The scale factor from ABC to DEF is $k = \frac{DE}{AB} = \frac{14}{10} = 1,4$.
- ③ We have $k > 1$, so triangle DEF is an enlargement of triangle ABC , with scale factor $k = 1,4$.

→ Exercise sheet: exercises 1.4 a), 2.1 a), 3.4 a) and 6.3 a) (model A1)

Why order the sides before comparing?

Multiplying by a number $k > 0$ preserves order: if $a \leq b \leq c$, then $k \times a \leq k \times b \leq k \times c$. If the sides of two triangles are proportional, the shortest side of one therefore corresponds to the shortest side of the other, the middle one to the middle one, and the longest to the longest. It is enough to test the ordered sides: if these three ratios are not equal, no other way of pairing the sides can give three equal ratios.

Writing model R2: recognising similar triangles from their sides

The steps

- ① Order the three sides of each triangle from shortest to longest, naming the sides.
- ② Calculate separately the three ratios of sides of the same rank, from the first triangle to the second: shortest with shortest, middle with middle, longest with longest.
- ③ Write the equality of the three ratios (or the inequality) and conclude: if equal, the side lengths are proportional and the triangles are similar, with scale factor k ; otherwise, they are not.

Sorting does not replace the proof: it only prevents pairing at random.

On your paper

In triangles ABC and DEF , we are given $AB = 14$, $BC = 8$, $AC = 12$ and $DE = 15$, $EF = 17,5$, $DF = 10$. Are these triangles similar? Justify using all three ratios.

- ① I order the sides of each triangle from shortest to longest: In ABC : $BC = 8$; $AC = 12$; $AB = 14$. In DEF : $DF = 10$; $DE = 15$; $EF = 17,5$.
- ② I calculate the three ratios of sides of the same rank, from ABC to DEF :

$$\frac{DF}{BC} = \frac{10}{8} = 1,25 \qquad \frac{DE}{AC} = \frac{15}{12} = 1,25 \qquad \frac{EF}{AB} = \frac{17,5}{14} = 1,25$$

- ③ We have $\frac{DF}{BC} = \frac{DE}{AC} = \frac{EF}{AB} = 1,25$: the three ratios are equal, the side lengths are proportional, so triangles ABC and EFD are similar ($A \leftrightarrow E$, $B \leftrightarrow F$, $C \leftrightarrow D$), with scale factor $k = 1,25$ from ABC to EFD (enlargement).

(If two ratios are different, write the inequality, for example $\frac{DF}{BC} \neq \frac{DE}{AC}$, and conclude that the triangles are not similar.)

▲ Warning

Two equal ratios are not enough when using the definition by the three sides. For example, with two triangles of sides 2, 5, 6 and 6, 15, 17:

$$\frac{6}{2} = \frac{15}{5} = 3 \quad \text{but} \quad \frac{17}{6} \neq 3, \quad \text{so} \quad \frac{6}{2} = \frac{15}{5} \neq \frac{17}{6}.$$

The triangles are not similar. The third ratio is not an optional check: it is part of the proof.

→ Exercise sheet: exercise 1.2 (model R2)

2.2 The angle characterisation

Theorem (characteristic property of similar triangles)

Two triangles are similar if, and only if, their angles are equal in measure in pairs.

In other words:

- if two triangles have their angles equal in measure in pairs, then their sides are proportional;
- conversely, if the sides of two triangles are proportional, then their angles are equal in measure in pairs.

The vertices of equal angles are corresponding; the sides opposite these angles are also corresponding.

It reads: “if, and only if”: the two implications above are true, and each is proved separately (below). In plain words: having the same shape means having angles of equal measure; and having angles of equal measure means having proportional sides.

A characterisation is a property that is equivalent to the definition: the object can be recognised by it. This theorem is the angle characterisation of similar triangles.

Two angles are enough

If two triangles have two pairs of equal angles, then the third angles are also equal, and the triangles are therefore similar.

Why are two angles enough?

Proof tool: direct proof, by calculation from a known property (the sum of the angles in a triangle).

In any triangle, the sum of the angles is 180° (proved in Year 8, chapter Angles in a triangle). If $\widehat{A} = \widehat{A'}$ and $\widehat{B} = \widehat{B'}$, then:

$$\widehat{C} = 180^\circ - \widehat{A} - \widehat{B} = 180^\circ - \widehat{A'} - \widehat{B'} = \widehat{C'}.$$

■

Why is the angle characterisation true?

Proof tool: an intermediate triangle AMN placed inside ABC , a congruence criterion (section 1.1) and proportionality in nested triangles (section 1.4). If $A'B' = AB$, a congruence criterion shows directly that the triangles are congruent; if $A'B' > AB$, we swap the roles of the two triangles; we therefore assume $A'B' < AB$, place M on $[AB]$ with $AM = A'B'$, and the line parallel to (BC) through M meets $[AC]$ at N .

Forward direction (from angles to sides). Suppose $\widehat{A} = \widehat{A'}$, $\widehat{B} = \widehat{B'}$ and $\widehat{C} = \widehat{C'}$. The angle $\widehat{MAN}$ is the angle $\widehat{BAC}$, since $M \in [AB]$ and $N \in [AC]$; hence $\widehat{MAN} = \widehat{BAC} = \widehat{A'}$. Since $(MN) \parallel (BC)$, corresponding angles give $\widehat{AMN} = \widehat{ABC} = \widehat{B'}$. With $AM = A'B'$, triangles AMN and $A'B'C'$ are congruent (“angle-side-angle” criterion), hence $AN = A'C'$ and $MN = B'C'$. The nested triangles then give $\frac{A'B'}{AB} = \frac{AM}{AB} = \frac{AN}{AC} = \frac{MN}{BC}$, so $\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$: the side lengths are proportional.

Converse direction (extension, revisited in exercise 7.5 of the exercise sheet). Suppose $\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC} = k$. Since $\frac{AM}{AB} = \frac{A'B'}{AB} = k$, proportionality in the nested triangles gives $\frac{AN}{AC} = \frac{MN}{BC} = k$, hence $AN = A'C'$ and $MN = B'C'$. Triangles AMN and $A'B'C'$ are then congruent (“side-side-side” criterion), so their angles are equal in pairs; now those of AMN are equal to those of ABC (common angle at A and corresponding angles). Therefore the angles of $A'B'C'$ are equal in pairs to those of ABC . ■

Writing model R1: recognising similar triangles from their angles

The steps

① Calculate the third angle of each triangle: 180° minus the sum of the two known angles.

② Write down the three angles of each triangle, from smallest to largest.

③ Compare and conclude using the angle characterisation: if the angles are equal in pairs, the triangles are similar; otherwise, they are not.

The conclusion “not similar” uses the other direction of the theorem: two similar triangles have angles equal in pairs.

On your paper

Triangle n° 1: two angles of 38° and 75° . Triangle n° 2: two angles of 67° and 38° . Are these triangles similar?

① In triangle n° 1, the third angle measures: $180^\circ - 38^\circ - 75^\circ = 67^\circ$.
In triangle n° 2, the third angle measures: $180^\circ - 67^\circ - 38^\circ = 75^\circ$.

② The angles of triangle n° 1 are 38° , 67° and 75° ; those of triangle n° 2 are 38° , 67° and 75° .

③ The two triangles have angles equal in pairs: by the angle characterisation, they are similar.

Your turn: same model

Triangle n° 1: two angles of 29° and 113° . Triangle n° 2: two angles of 38° and 29° . Are these triangles similar?

Answer: in triangle n° 1, the third angle measures $180^\circ - 29^\circ - 113^\circ = 38^\circ$ and in triangle n° 2, it measures $180^\circ - 38^\circ - 29^\circ = 113^\circ$; both triangles have the angles 29° , 38° and 113° ; by the angle characterisation, they are similar.

→ Exercise sheet: exercises 1.1, 3.8 d) and 6.4 a) (model R1)

Same shape, letters in a different order

We are given:

	A	B	C		D	E	F
triangle ABC	85°	58°	37°	triangle DEF	58°	37°	85°

The corresponding vertices are those with equal angles:

$$A \leftrightarrow F, \quad B \leftrightarrow D, \quad C \leftrightarrow E.$$

Thus:

$$[AB] \leftrightarrow [FD], \quad [BC] \leftrightarrow [DE], \quad [CA] \leftrightarrow [EF].$$

If $AB = 5$ and $FD = 8$, then the scale factor from triangle ABC to triangle FDE is $k = \frac{FD}{AB} = \frac{8}{5} = 1,6$; we deduce $DE = 1,6 \times BC$ and $EF = 1,6 \times CA$.

Checking by order. In each triangle, the largest angle is 85° (vertices A and F): the opposite sides $[BC]$ and $[DE]$ are indeed matched together.

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Why must we test the position of the given angle in an isosceles triangle?

In an isosceles triangle, the two base angles are equal (part 1), whereas the apex angle may be different. A given angle can therefore be either the apex angle or a base angle: these give two different triangles, and the angle sum, 180° , determines the other two angles in each case. A position is impossible when the sum of the two base angles would reach or exceed 180° .

Writing model R3: finding the angles of an isosceles triangle given one angle

The steps

- ① Place the given angle at the apex: the two equal base angles each measure 180° minus this angle, divided by 2.
- ② Place the given angle at the base: the other base angle has the same measure, and the apex angle is 180° minus their sum; if this sum reaches or exceeds 180° , this case is impossible.
- ③ Conclude: if only one case is possible, all three angles are fixed; if two cases are possible, there are two different shapes.

On your paper

Two isosceles triangles each have an angle of 70° .

a) Are these triangles necessarily similar?

b) Same question with an angle of 110° .

a) ① If the 70° angle is the apex angle, the two base angles each measure $(180^\circ - 70^\circ) \div 2 = 110^\circ \div 2 = 55^\circ$: the angles are 70° , 55° and 55° .

② If the 70° angle is a base angle, the other base angle also measures 70° , and the apex angle is $180^\circ - 70^\circ - 70^\circ = 40^\circ$: the angles are 70° , 70° and 40° .

③ Two cases are possible, with different angles: these triangles are not necessarily similar.

b) ① If the 110° angle is the apex angle, the two base angles each measure $(180^\circ - 110^\circ) \div 2 = 70^\circ \div 2 = 35^\circ$: the angles are 110° , 35° and 35° .

② If the 110° angle were a base angle, the other base angle would also measure 110° : these two angles would already add up to 220° , more than 180° . This case is impossible.

③ Only one case is possible: the angles are necessarily 110° , 35° and 35° . Two such triangles have angles equal in pairs: by the angle characterisation, they are always similar.

→ Exercise sheet: exercises 1.1 e), 1.5 d) and 1.5 e) (model R3)

Writing model D2: proving that two triangles are similar (two pairs of angles)

The steps

- ① We know that: write a first pair of equal angles, with its justification (given in the problem, common angle, vertically opposite angles, angles formed by parallel lines, right angles, previous question).
- ② Similarly, we know that: write a second pair of equal angles, with its justification.
- ③ Now: state the property: if two triangles have two pairs of equal angles, then they are similar.
- ④ Therefore: conclude by naming the two triangles and the three pairs of corresponding vertices (the vertices of the equal angles).

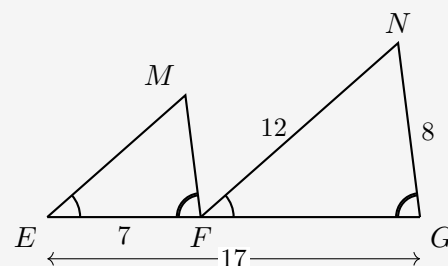
This is the standard proof layout for the chapter. The justification sentences are listed in the box below.

Question 1 of this example is written up using model D1 (section 1.3).

On your paper

The points E, F, G are collinear in this order; $(EM) \parallel (FN)$ and $(FM) \parallel (GN)$.

1. Prove that $\widehat{FEM} = \widehat{GFN}$.
2. Prove that triangles EFM and FGN are similar.



2. ① We know that $\widehat{FEM} = \widehat{GFN}$, from question 1.
- ② Similarly, we know that $\widehat{EFM} = \widehat{FGN}$, as they are corresponding angles formed by the parallel lines (FM) and (GN) with the transversal (EG) .
- ③ Now, if two triangles have two pairs of equal angles, then they are similar.
- ④ Therefore triangles EFM and FGN are similar, with corresponding vertices $E \leftrightarrow F, F \leftrightarrow G$ and $M \leftrightarrow N$. ■

Justifying a pair of equal angles: one sentence per source

In a pair, the angle from the first triangle named in the problem is written to the left of the $=$ sign.

- Given in the problem: “Similarly, we know that $\widehat{ADE} = \widehat{ACB}$, as this is given in the problem.”
- Common angle: “The angle at A is common to triangles ADE and ACB (with $D \in [AB]$ and $E \in [AC]$), so $\widehat{DAE} = \widehat{CAB}$.” (On rays: “with $A' \in [CA)$ and $B' \in [CB)$.”)
- Vertically opposite angles: “The angles $\widehat{AOB}$ and $\widehat{COD}$ are vertically opposite (with the lines (AC) and (BD) intersecting at O), so $\widehat{AOB} = \widehat{COD}$.”
- Angles formed by parallel lines: “Similarly, we know that $\widehat{OAB} = \widehat{OCD}$, as they are alternate interior angles formed by the parallel lines (AB) and (CD) with the transversal (AC) .” (or “corresponding”, depending on their position)
- Right angles (altitudes configuration, part 3.3): “Similarly, we know that $\widehat{ABE} = \widehat{DCE} = 90^\circ$, as both angles are right angles: $(AB) \perp (BD)$ and $(CD) \perp (AC)$.”

In the conclusion, the second triangle is named in the order of correspondence: if D , on $[AB]$, corresponds to C , we write “Therefore triangles ADE and ACB are similar, with corresponding vertices $A \leftrightarrow A, D \leftrightarrow C$ and $E \leftrightarrow B$. ■”

→ Exercise sheet: exercises 1.3, 1.4 a), 2.2 b), 2.3 a) and d), 2.4 c) and d), 3.3 a), 3.4 a), 5.3 a), 6.1 b) and 6.2 a) (model D2)

▲ Warning

- A single pair of equal angles is not enough. Triangles with angles 40° – 110° – 30° and 40° – 70° – 70° share a 40° angle and are not similar. Two pairs of equal angles are required.
- Similar does not mean congruent. Two similar triangles have the same shape; they only have the same size if $k = 1$.
- On a drawing, “it looks the same” proves nothing: always justify using the angles or the three ratios, never by eye.

Counterexample

A general statement says that something is true for all objects of a certain type, for example “two isosceles triangles that each have an angle of 70° are always similar”. A counterexample is a case that satisfies the hypothesis of the statement without satisfying its conclusion: here, two isosceles triangles with angles $70^\circ, 55^\circ, 55^\circ$ and $70^\circ, 70^\circ, 40^\circ$ (model R3). A single counterexample is enough to prove that a general statement is false.

Why are examples not enough to answer “true”?

An “always true” statement applies to infinitely many triangles: even a hundred examples that satisfy it leave all the others aside. To answer “true”, we need an argument that holds for all the triangles concerned (a definition or a property from the course notes). To answer “false”, a single case that fails is enough, provided it is given with specific numbers that can be checked.

Writing model T2: answering a “true or false” question

The steps

- ① State the answer: true or false.
- ② True statement: prove it for all the triangles concerned (from the definition, or by showing that their angles are fixed, then using the angle characterisation).
- ③ False statement: give a numerical counterexample, two triangles that satisfy the hypothesis without satisfying the conclusion.

On your paper

True or false? Justify.

- a) Two right-angled triangles that each have an angle of 35° are always similar.
- b) Two triangles that each have a side of 5 cm are always similar.

a) ① TRUE.

② In a right-angled triangle that has an angle of 35° , this angle is not the right angle, and the third angle is $180^\circ - 90^\circ - 35^\circ = 55^\circ$. Two such triangles therefore have angles equal in pairs, 35° , 55° and 90° : by the angle characterisation, they are always similar. ■

b) ① FALSE.

③ Counterexample: an equilateral triangle of side 5 cm, whose three angles measure 60° , and an isosceles right-angled triangle whose legs measure 5 cm, and whose angles measure 90° , 45° and 45° . They each have a side of 5 cm, but do not have equal angles: they are not similar.

→ Exercise sheet: exercise 1.5 (model T2; in d) and e), model R3)

Quick questions (unaided)

1. One triangle has angles of 52° and 76° ; another has angles of 52° and 52° . Are they similar?
2. Two similar triangles have scale factor $k = 3$. How do the angles of the second compare with those of the first?
3. Do the sides 4; 7; 9 and 8; 14; 17 define similar triangles?

Answers: 1) yes! The first has third angle $180^\circ - 52^\circ - 76^\circ = 52^\circ$, the second $180^\circ - 52^\circ - 52^\circ = 76^\circ$: in both cases, the angles are 52° , 76° ; 2) exactly the same, the angles never change, only k acts on the lengths; 3) no: $\frac{4}{8} = \frac{7}{14} = 2$, $\frac{9}{17} \neq 2$.

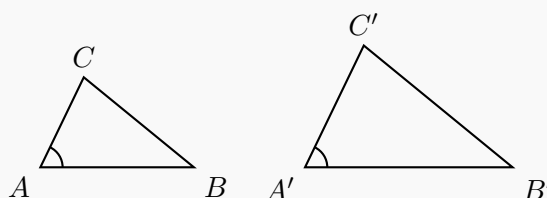
2.3 A third criterion: “SAS for similarity”

Two angles are enough, provided we actually know two angles. When the problem only gives one angle and lengths, the following criterion takes over.

Third similarity criterion (assumed): an angle included between two proportional sides

If two triangles have an equal angle included between two pairs of proportional sides, then they are similar.

In other words: if $\widehat{BAC} = \widehat{B'A'C'}$ and $\frac{A'B'}{AB} = \frac{A'C'}{AC}$, then triangles ABC and $A'B'C'$ are similar.



Why is this criterion true? (idea of the proof)

This criterion is to similar triangles what the “side-angle-side” (or “SAS” for short) congruence criterion is to congruent triangles: “sides of equal length in pairs” is replaced by “sides proportional in pairs”. Just as for the congruence criterion, the angle must be included between the two sides.

Its proof relies on a congruence criterion and on proportionality in nested triangles. If $A'B' = AB$, then $\frac{A'C'}{AC} = \frac{A'B'}{AB} = 1$, so $A'C' = AC$ and the “side-angle-side” criterion shows directly that the triangles are congruent; if $A'B' > AB$, we swap the roles of the two triangles. If $A'B' < AB$, we place M on $[AB]$ with $AM = A'B'$, and the line parallel to (BC) through M meets $[AC]$ at N . The nested triangles give $\frac{AN}{AC} = \frac{AM}{AB} = \frac{A'B'}{AB} = \frac{A'C'}{AC}$, so $AN = A'C'$. Triangles AMN and $A'B'C'$ are then congruent (“side-angle-side” criterion); now the angles of AMN are equal to those of ABC (common angle and corresponding angles), so those of $A'B'C'$ are too: by the angle characterisation, ABC and $A'B'C'$ are similar.

Writing model D3: proving that two triangles are similar (an angle included between two proportional sides)

The steps

① We know that: write the pair of equal angles, with its justification.

② Calculate the ratios of the two pairs of sides enclosing these angles, the smaller with the smaller, and note that they are equal.

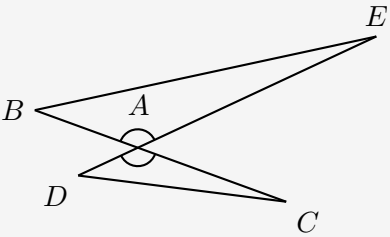
③ Now: state the third criterion: if two triangles have an equal angle included between two pairs of proportional sides, then they are similar.

④ Therefore: conclude by naming the two triangles and the three pairs of corresponding vertices.

Only one pair of equal angles is known: the angle characterisation cannot conclude on its own.

On your paper

The segments $[BC]$ and $[DE]$ intersect internally at A ;
 $AD = 15$ cm, $AB = 25$ cm,
 $AC = 36$ cm and $AE = 60$ cm; no parallel lines are given. Prove that triangles DAC and BAE are similar.



① The angles $\widehat{DAC}$ and $\widehat{BAE}$ are vertically opposite (with the lines (DE) and (BC) intersecting at A), so $\widehat{DAC} = \widehat{BAE}$.

② The sides enclosing these angles measure $AD = 15$ cm and $AC = 36$ cm in triangle DAC , $AB = 25$ cm and $AE = 60$ cm in triangle BAE :

$$\frac{AB}{AD} = \frac{25}{15} = \frac{5}{3}$$

$$\frac{AE}{AC} = \frac{60}{36} = \frac{5}{3}$$

The two ratios are equal.

③ Now, if two triangles have an equal angle included between two pairs of proportional sides, then they are similar.

④ Therefore triangles DAC and BAE are similar, with corresponding vertices $D \leftrightarrow B$, $A \leftrightarrow A$ and $C \leftrightarrow E$. ■

→ Exercise sheet: exercise 2.5 a) (model D3)

Quick questions (unaided)

1. $\widehat{A} = \widehat{A'} = 41^\circ$, $AB = 6$, $AC = 9$, $A'B' = 8$, $A'C' = 12$: are triangles ABC and $A'B'C'$ similar?

2. Same question with $A'B' = 8$ and $A'C' = 10$.

Answers: 1) yes: $\frac{6}{8} = \frac{9}{12} = \frac{3}{4}$, equal angle included between proportional sides (SAS for similarity); 2) no: $\frac{6}{10} \neq \frac{9}{12}$ and $\frac{6}{8} \neq \frac{9}{10}$, no way of matching the sides around the equal angles gives equal ratios.

3 Calculating with similar triangles

Knowing that two triangles are similar is good; using this to calculate lengths is the heart of the exercises in this chapter. The whole difficulty comes down to one word: matching.

Starting triangle, target triangle

To calculate with two similar triangles, we choose a direction: the scale factor k goes from the starting triangle to the target triangle. Each length in the target triangle is k times the corresponding length in the starting triangle: $A'B' = k \times AB$.

To go back to the starting triangle, we divide by k : from $A'B' = k \times AB$, we get $AB = A'B' \div k$.

In plain words: with $k = 3$, a starting side of 7 cm gives a target side of $3 \times 7 = 21$ cm, and a target side of 12 cm comes from a starting side of $12 \div 3 = 4$ cm.

3.1 Calculating lengths: the model example

We return to the model example from section 1.3. Question 1 (model D1) showed that $\widehat{FEM} = \widehat{GFN}$, and question 2 (model D2, section 2.2) showed that triangles EFM and FGN are similar, with $E \leftrightarrow F$, $F \leftrightarrow G$ and $M \leftrightarrow N$. Question 3 remains: deduce the lengths EM and FM .

Why match, set up, then solve?

The scale factor k is only the same for corresponding sides: dividing two sides that do not correspond gives a meaningless number. We therefore match them first, using the corresponding vertices, never “by eye”: the drawing can be misleading. The table places each side under its corresponding side; one complete column is enough to calculate k , since all ratios of corresponding sides equal k . We then multiply by k to go to the target triangle, and divide by k to go back to the starting triangle. The two checks at the end catch any error in matching or direction: if $k < 1$, each side of the target triangle must be shorter than its corresponding side.

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Writing model L1: calculating lengths in two similar triangles (match, set up, solve)

The steps

① Match: write down the corresponding sides, which join two corresponding vertices (never “by eye”).

② Set up: arrange the corresponding sides in a table, starting triangle on the top row, target triangle on the bottom row; if there is no complete column, first calculate the missing length.

③ Solve: calculate the scale factor k from a complete column: target length divided by the corresponding starting length.

④ Calculate each required length: multiply by k to go to the target triangle, divide by k to go back to the starting triangle.

⑤ Check: the ratios of the complete columns are equal, and the smaller triangle has the shorter sides.

The starting triangle is FGN : it is the one with the most known sides, once FG has been calculated.

On your paper

The points E, F, G are collinear in this order; triangles EFM and FGN are similar (question 2); $EF = 7$, $EG = 17$, $FN = 12$ and $GN = 8$.

3. Deduce the lengths EM and FM .

3. ① From question 2, $E \leftrightarrow F$, $F \leftrightarrow G$ and $M \leftrightarrow N$: the corresponding sides are $[EF] \leftrightarrow [FG]$, $[EM] \leftrightarrow [FN]$ and $[FM] \leftrightarrow [GN]$.

② The point F lies on the segment $[EG]$, so $FG = EG - EF = 17 - 7 = 10$. I arrange the corresponding sides in a table:

sides of FGN	$FG = 10$	$FN = 12$	$GN = 8$
sides of EFM	$EF = 7$	$EM = ?$	$FM = ?$

③ The scale factor from FGN to EFM is $k = \frac{EF}{FG} = \frac{7}{10} = 0,7$ (reduction).

④ To go to triangle EFM , I multiply by k :

$EM = 0,7 \times FN = 0,7 \times 12 = 8,4$

$FM = 0,7 \times GN = 0,7 \times 8 = 5,6$

⑤ Check: $\frac{7}{10} = \frac{8,4}{12} = \frac{5,6}{8} = 0,7$: the ratios are equal. The scale factor is strictly less than 1, so each side of EFM must be strictly shorter than its corresponding side in FGN : this is the case, $7 < 10$, $8,4 < 12$ and $5,6 < 8$. ✓

From reasoning to equal ratios

We only write the equality of ratios after justifying that the triangles are similar and matching the sides using the angles. The word “therefore” links the geometric argument to proportionality; each subsequent = sign preserves the same value. A measurement taken from the drawing is written with the symbol $\approx$ (“approximately equal to”) as long as no theorem guarantees exact equality.

Equality of cross-products

For numbers a, b, c, d , with $b \neq 0$ and $d \neq 0$:

$$\frac{a}{b} = \frac{c}{d} \text{ is equivalent to } a \times d = b \times c.$$

It reads: “ a over b equals c over d ”. In plain words: two quotients are equal precisely when the “cross-products” $a \times d$ and $b \times c$ are equal; for example, $\frac{2}{5} = \frac{6}{15}$ and $2 \times 15 = 30 = 5 \times 6$.

Why is this true?

We multiply both sides of $\frac{a}{b} = \frac{c}{d}$ by the same non-zero number $b \times d$: we obtain $a \times d$ on the left and $c \times b$ on the right, so $a \times d = b \times c$. Conversely, we divide both sides of $a \times d = b \times c$ by $b \times d$. ■

Two methods for the “solve” step: which one to choose?

To calculate EM in the model example (from FGN to EFM):

• Method 1 (using the scale factor): $k = \frac{7}{10} = 0,7$, then $EM = 0,7 \times 12 = 8,4$. Quick when k is a simple number.

• Method 2 (cross-multiplication): in the table, $\frac{EM}{12} = \frac{7}{10}$, so $10 \times EM = 7 \times 12$ and $EM = \frac{7 \times 12}{10} = 8,4$. Always available, even when k is a complicated fraction.

Both methods are valid in any proportionality table. On your paper, always calculate using the scale factor k (step ④ of model L1), which will be used again for the check and for areas; only add cross-multiplication if the problem asks for it. Both will be used again in chapter 01C.

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→ Exercise sheet: exercises 1.4 b) and c), 2.1 b) and c), 2.2 c), 2.3, 2.4 c) and d), 5.1 b), 5.3, 6.1 c), 6.2 b), 6.4 b) and c) (model L1, after model D2 when similarity must be proved first)

Altitude, median

The altitude from a vertex of a triangle is the line passing through that vertex and perpendicular to the opposite side; the segment joining the vertex to the foot of this perpendicular is also called the altitude, as is its length. The median from a vertex is the segment joining that vertex to the midpoint of the opposite side; its length is also called the median.

The scale factor k acts on all corresponding lengths

If two triangles are similar with scale factor k , then:

- their corresponding sides are in the ratio k ;
- their perimeters are in the ratio k ;
- their corresponding altitudes and their corresponding medians are in the ratio k .

In other words, any length constructed in the same way in both triangles is multiplied by k .

Same scale factor, several quantities

A triangle T' is an enlargement of a triangle T with scale factor $k = 1,5$. The perimeter of T is 24 cm, an altitude measures 4,4 cm and the median from the same vertex measures 6,2 cm. Then:

$$P' = 1,5 \times 24 = 36 \text{ cm}, \quad h' = 1,5 \times 4,4 = 6,6 \text{ cm}, \quad m' = 1,5 \times 6,2 = 9,3 \text{ cm}.$$

The reflex is the same in all three calculations: these are lengths, so they are multiplied by k , not by k^2 .

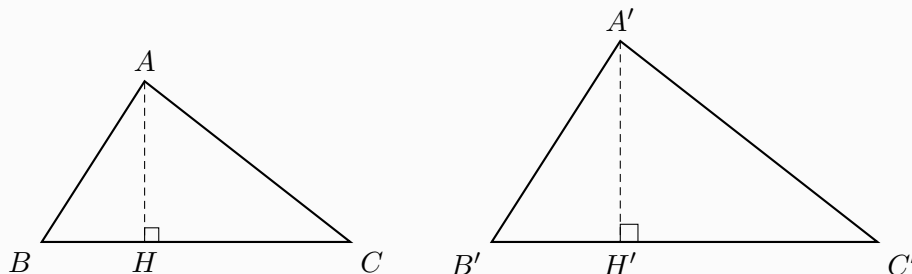
Why is an altitude also multiplied by k ?

Proof tool: direct proof, with two cases depending on the position of the foot of the altitude.

Suppose ABC and $A'B'C'$ are similar with

$$A \leftrightarrow A', \quad B \leftrightarrow B', \quad C \leftrightarrow C',$$

and draw the altitudes $[AH]$ and $[A'H']$ from the corresponding vertices A and A' .



If $\widehat{B} = 90^\circ$, then $H = B$ and $H' = B'$: the altitudes are the sides $[AB]$ and $[A'B']$, so the ratio of their lengths is k . Otherwise, triangles ABH and $A'B'H'$ are right-angled at H and H' . If $\widehat{B}$ is acute, H lies on the ray $[BC)$, possibly beyond C ; the angles at B and B' are the original corresponding angles. If it is obtuse, the angles at B and B' of the right-angled triangles are $180^\circ - \widehat{B} = 180^\circ - \widehat{B}'$. In both cases, the right-angled triangles have two pairs of equal angles: they are similar. Thus

$$\frac{A'H'}{AH} = \frac{A'B'}{AB} = k.$$

For the medians $[AM]$ and $[A'M']$, M and M' are the midpoints of $[BC]$ and $[B'C']$. Then $B'M'/BM = B'C'/BC = k = A'B'/AB$, and the included angles at B and B' are equal. The SAS similarity criterion (section 2.3) gives $ABM \sim A'B'M'$, hence $A'M'/AM = k$.

For the perimeters, the calculation is direct:

$$P' = kAB + kBC + kCA = k(AB + BC + CA) = kP.$$

→ Exercise sheet: exercise 2.1 d) (perimeters)

Writing model V2: finding and correcting an error in a pupil's work

The steps

- ① Re-read the work in the order of the steps: match, set up, solve.
- ② State the incorrect step and explain the error in one sentence, using the rule from the course notes.
- ③ Write out the correct solution following the steps for calculating lengths, then check.

We re-read in the order of the steps, because everything that follows a wrong step is wrong too: we must find the first error.

In b), step ③ is the full written solution from model L1: the numbered steps in b) are those of L1.

On your paper

Triangles ABC and DEF are similar, with $\hat{A} = \hat{D}$ and $\hat{C} = \hat{F}$; $AB = 6$, $AC = 8$ and $DF = 10$. To calculate DE , Hugo writes: " $\frac{DF}{AB} = \frac{10}{6} = \frac{5}{3}$, so $DE = \frac{5}{3} \times AC = \frac{5}{3} \times 8 = \frac{40}{3}$."

a) Find the error. b) Write out the correct solution.

a) ① I re-read the work in the order of the steps: Hugo calculates a ratio without matching the sides first.

② The error is at the "match" step: Hugo divided DF by AB , which are not corresponding sides. Yet corresponding sides join corresponding vertices: with $A \leftrightarrow D$ and $C \leftrightarrow F$, the side corresponding to $[DF]$ is $[AC]$, and the side corresponding to $[DE]$ is $[AB]$.

b) ① The angles give $A \leftrightarrow D$ and $C \leftrightarrow F$, and therefore also $B \leftrightarrow E$, the third angles: the corresponding sides are $[AB] \leftrightarrow [DE]$, $[AC] \leftrightarrow [DF]$ and $[BC] \leftrightarrow [EF]$.

② I arrange the corresponding sides in a table:

sides of ABC	$AB = 6$	$AC = 8$	BC
sides of DEF	$DE = ?$	$DF = 10$	EF

③ The scale factor from ABC to DEF is $k = \frac{DF}{AC} = \frac{10}{8} = 1,25$.

④ To go to triangle DEF , I multiply by k :

$$DE = 1,25 \times AB = 1,25 \times 6 = \boxed{7,5}$$

⑤ Check: $\frac{10}{8} = \frac{7,5}{6} = 1,25$: the ratios are equal. The scale factor is strictly greater than 1, so each side of DEF must be strictly longer than its corresponding side in ABC : this is the case, $10 > 8$ and $7,5 > 6$. ✓

→ Exercise sheet: exercise 5.1 (model V2, then model L1 in b))

3.2 The bow-tie configuration

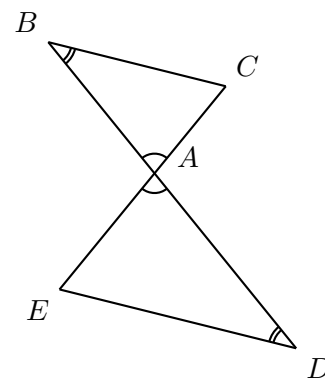
A configuration to know: the bow-tie

Segments $[BD]$ and $[CE]$ intersect internally at A , and lines (BC) and (DE) are parallel.

Why are triangles ABC and ADE similar? Two pairs of equal angles:

- $\widehat{BAC} = \widehat{DAE}$: vertically opposite angles;
- $\widehat{ABC} = \widehat{ADE}$: alternate interior angles formed by the parallel lines (BC) and (DE) with the transversal (BD) .

Therefore ABC and ADE are similar ($A \leftrightarrow A$, $B \leftrightarrow D$ and $C \leftrightarrow E$), and we calculate as usual: match, set up, solve.



$(BC) \parallel (DE)$

This figure is called the "bow-tie" (or "butterfly") configuration. In chapter 01C, the intercept theorem (Thales' theorem in France) will give direct access to it; for now, angles are more than enough.

Your turn: faded worked example to complete, from matching to self-checking

In the bow-tie figure above, we are given: $AB = 3$; $AD = 4,5$; $AC = 2,2$; $DE = 6$. Support is gradually removed at each stage: only read the next stage after finishing the previous one.

Stage 1: match (template provided). With $A \leftrightarrow A$, $B \leftrightarrow D$ and $C \leftrightarrow E$, the corresponding sides are $[AB] \leftrightarrow [\dots]$, $[AC] \leftrightarrow [\dots]$, $[BC] \leftrightarrow [\dots]$.

Stage 2: set up. Copy and complete the table of corresponding sides, with the starting triangle ABC in the top row and the target triangle ADE in the bottom row.

Stage 3: solve without a given formula. Calculate the scale factor k from ABC to ADE , then AE and BC .

Review, independent check. The three ratios $\frac{AD}{AB}$, $\frac{AE}{AC}$, $\frac{DE}{BC}$ are all equal to; the lengths are consistent, because

Turn over the answer only after checking: corresponding sides $[AD]$, $[AE]$, $[DE]$: $k = \frac{4,5}{3} = 1,5$; $AE = 1,5 \times 2,2 = 3,3$; $BC = 6 \div 1,5 = 4$; common ratio 1,5. We have $k > 1$, so each side of ADE is strictly longer than its corresponding side in ABC .

Writing model J1: proving a property (we know that, now, therefore)

The steps

① We know that: write down the useful hypotheses, taken from the problem or from a previous question.

② Now: state in full the property, definition or calculation rule that applies.

③ Therefore: write what it gives on the figure.

④ Chain together if needed: the conclusion obtained becomes a known fact; repeat until the required result is reached, then finish with ■.

Similar triangles give equal ratios, and equal ratios give equal products (cross-multiplication, section 3.1): step ④ chains this second step, the conclusion of the first becoming a known fact.

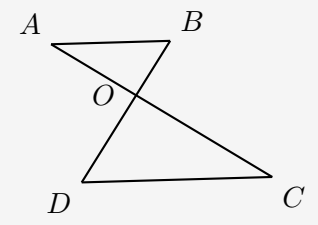
Question a) is written using model D2; the solution opposite is for question b).

On your paper

Segments $[AC]$ and $[BD]$ intersect at O , and lines (AB) and (CD) are parallel.

a) Prove that triangles OAB and OCD are similar.

b) Deduce that $OA \times OD = OB \times OC$.



$(AB) \parallel (CD)$

b) ① We know that triangles OAB and OCD are similar, the corresponding vertices being $O \leftrightarrow O$, $A \leftrightarrow C$ and $B \leftrightarrow D$, from question a).

② Now, if two triangles are similar, then the lengths of their corresponding sides are proportional.

③ Therefore $\frac{OC}{OA} = \frac{OD}{OB}$.

④ Now, if $\frac{a}{b} = \frac{c}{d}$, with b and d non-zero, then $a \times d = b \times c$: this is the cross-multiplication property. Therefore $OC \times OB = OA \times OD$, that is, $OA \times OD = OB \times OC$. ■

→ Exercise sheet: exercise 4.5 a) and b) 🧑 optional (model J1)

Quick questions (unaided)

1. In two similar triangles, opposite equal angles, what do we find?

2. ABC and DEF are similar ($A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$), with $AB = 5$, $DE = 8$. What is the scale factor from ABC to DEF ? And what is EF if $BC = 4$?

3. What is the first step of model L1, and why not another one?

Answers: 1) corresponding sides; 2) $k = \frac{8}{5} = 1,6$, so $EF = 1,6 \times 4 = 6,4$; 3) match: until the corresponding sides are identified using the angles, any table and any calculation may be wrong.

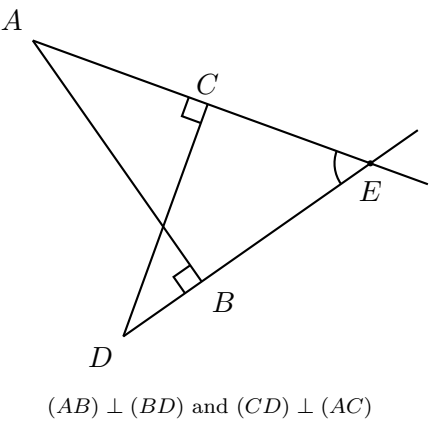
→ Exercise sheet: exercises 1.4 and 6.2 (bow-tie configuration: models D2, A1 and L1)

3.3 A second configuration to recognise: intersecting altitudes

“Altitudes” configuration

If lines (AC) and (BD) intersect at E , and if $(AB) \perp (BD)$ and $(CD) \perp (AC)$, then triangles ABE and DCE , assumed non-degenerate, are similar.

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Why? (two lines are enough)

Proof tool: direct proof (two pairs of equal angles).

The angle at E is shared by triangles ABE and DCE (or vertically opposite in another layout). Each also has a right angle: $\widehat{ABE} = 90^\circ$ because $(AB) \perp (BD)$, and $\widehat{DCE} = 90^\circ$ because $(CD) \perp (AC)$. Two pairs of equal angles: triangles ABE and DCE are similar, with $B \leftrightarrow C$ (the right angles), $E \leftrightarrow E$, $A \leftrightarrow D$. ■

Remark

Hence the name: in triangle ADE , (AB) is the altitude from A (it is perpendicular to the opposite side (ED)) and (DC) is the altitude from D . Two intersecting altitudes therefore always form similar triangles: a configuration to remember alongside the bow-tie. It appears again on the exercise sheet (exercise 4.5) and in many problems involving circles and tangents.

Quick questions (unaided)

1. In the altitudes configuration above, which angle is shared by both triangles?

2. Which are the corresponding vertices, and why?

Answers: 1) the angle at E , formed by the two intersecting lines; 2) $B \leftrightarrow C$ (the two right angles), $E \leftrightarrow E$ (the shared angle), $A \leftrightarrow D$ (the third angles), always by the angles, never by position.

→ Exercise sheet: exercise 4.5 optional (altitudes configuration)

4 Enlargement and reduction

Enlarging is not stretching

We start with a right-angled isosceles triangle whose two legs each measure 4 cm.

A original: 4 by 4

D transformation 1: 6 by 6

G transformation 2: 6 by 4

1. For each transformation, calculate the factor applied to the horizontal side, then the factor applied to the vertical side.

2. Which one preserves the 45° angle of the initial triangle? Which one distorts the triangle?

3. State the condition that distinguishes a true enlargement from a simple stretch.

Summary. In transformation 1, both factors equal $\frac{6}{4} = 1,5$: all lengths are enlarged by the same scale factor and the angles are preserved. In transformation 2, the factors are $\frac{6}{4} = 1,5$ horizontally but $\frac{4}{4} = 1$ vertically: there is no single scale factor. The acute

angle at H cannot measure 45° : a right-angled triangle with a 45° angle would be isosceles, whereas here the two legs measure 6 and 4. The triangle is stretched, not enlarged in the mathematical sense.

4.1 Definition and scale

Enlargement, reduction

Let $k > 0$. A figure $\mathcal{F}'$ is an enlargement (if $k > 1$) or a reduction (if $0 < k < 1$) of a figure $\mathcal{F}$ with scale factor k when all the lengths of $\mathcal{F}'$ are obtained by multiplying those of $\mathcal{F}$ by k .

It reads: $\mathcal{F}$ “script F” and $\mathcal{F}'$ “script F prime”: the script letter denotes any figure (triangle, rectangle, disc, solid) and the prime indicates the transformed figure. In plain words: enlarging with scale factor 3 means tripling all the lengths of the figure; reducing with scale factor $\frac{1}{2}$ means dividing them all by 2. The case $k = 1$ gives back the starting figure, up to a rigid motion.

Remark

- For triangles, “ T' is an enlargement or a reduction of T ” is equivalent to “ T' and T are similar, with $k \neq 1$ ”: it is the same scale factor k . The case $k = 1$ is that of congruent triangles.
- The scale of a plan or a map is the scale factor of the reduction, from reality to the plan: a plan at a scale of $\frac{1}{250}$ means $k = \frac{1}{250}$, that is, 1 cm on the plan represents 250 cm = 2,5 m in reality.
- Watch the vocabulary: a scale factor of 130 % ($k = 1,3 > 1$) is an enlargement; a reduction has a scale factor strictly between 0 and 1 (for example 80 % = 0,8).

Why convert before dividing?

A scale compares two lengths: it is a unitless quotient, which must not depend on the chosen unit. On a plan, if 4 cm represent 1 m, the quotient $\frac{4}{1} = 4$ would suggest an enlargement; by writing 1 m = 100 cm, we obtain $\frac{4}{100} = \frac{1}{25}$, a reduction.

Writing model A1: calculating a scale factor k (similarity, enlargement, reduction) or a scale

The steps ① Choose two known corresponding lengths and write them in the same unit. ② Divide the target length by the corresponding starting length: this is k; simplify. ③ Conclude: if $k > 1$, it is an enlargement; if $0 < k < 1$, a reduction; for a plan or a map, the scale is written $\frac{1}{n}$.	On your paper On a plan, a 12,5 m wall is represented by a 5 cm segment. What is the scale of the plan? ① The corresponding lengths are 12,5 m in reality and 5 cm on the plan; in the same unit, 12,5 m = 1 250 cm. ② The scale factor from reality to the plan is $k = \frac{5}{1\,250} = \frac{1}{250}$. ③ We have $0 < k < 1$, so it is a reduction: the scale of the plan is $\frac{1}{250}$.
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→ Exercise sheet: exercises 3.1 f) and g), 3.1 bis a), 3.5 a) and 4.1 b) (model A1)

4.2 The effects of an enlargement or reduction: k , k^2 , k^3

Effects on lengths, angles, areas and volumes

In an enlargement or a reduction with scale factor k :

lengths (sides, perimeters, diagonals. . .)	$\times k$	1 dimension
angle measures	preserved	the shape does not change
areas	$\times k^2$	2 dimensions
volumes	$\times k^3$	3 dimensions

It reads: k^2 “ k squared” ($k \times k$) and k^3 “ k cubed” ($k \times k \times k$). In plain words: if lengths are multiplied by 4 ($k = 4$), areas are multiplied by 16 and volumes by 64; if they are divided by 10 ($k = \frac{1}{10}$), areas are divided by 100 and volumes by 1 000.

Why are areas multiplied by k^2 ?

Proof tool: direct calculation on a rectangle, then extension to any surface.

On a rectangle. Consider a rectangle of length L and width ℓ , with area $\mathcal{A} = L \times \ell$. Its enlargement with scale factor k is a rectangle of dimensions kL and $k\ell$, and therefore with area:

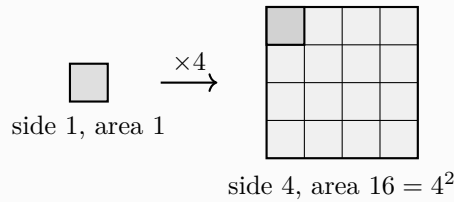
$$\mathcal{A}' = (kL) \times (k\ell) = k \times k \times L \times \ell = k^2 \times \mathcal{A}.$$

On a triangle. If a base measures b and the corresponding altitude h , the area is $\mathcal{A} = \frac{b \times h}{2}$. In the similar triangle, the base and the altitude are both corresponding lengths: they become kb and kh . Therefore

$$\mathcal{A}' = \frac{(kb) \times (kh)}{2} = k^2 \times \frac{b \times h}{2} = k^2 \mathcal{A}.$$

The second factor k comes from the altitude, even when it is not drawn.

We can see this on a square. A square of side 1 enlarged with scale factor 4 becomes a square of side 4... which contains $16 = 4^2$ unit squares:



For the standard shapes studied in lower secondary school, the area can be approximated using finer and finer grids. This suggests extending the $\times k^2$ rule, proved above for rectangles and triangles; its extension to other standard shapes is assumed here. Similarly, a cuboid with dimensions L, ℓ, h , when enlarged, becomes a cuboid $kL, k\ell, kh$, with volume $k^3 \times L\ell h$: volumes are multiplied by k^3 .

▲ Warning

The costliest mistake in this chapter: the general rule for areas is multiplication by k^2 . The expressions k and $2k$ do not give this rule, even though they may coincide with k^2 for certain values. Multiplying lengths by 10 means multiplying the area by 100 and the volume by 1000. Before concluding, ask yourself: is my quantity a length (1D), an area (2D), or a volume (3D)?

Why k^2 ? A plot of land on a map, two calculation methods that agree

On a map at a scale of $\frac{1}{25\,000}$, a rectangular plot measures 4,4 cm by 3,2 cm.
Method 1: go through the lengths.

$$4,4 \times 25\,000 = 110\,000 \text{ cm} = 1,1 \text{ km}, \quad 3,2 \times 25\,000 = 80\,000 \text{ cm} = 0,8 \text{ km}.$$

The actual area is therefore $1,1 \times 0,8 = 0,88 \text{ km}^2$.

Method 2: use the ratio of the areas directly. The area on the map is $4,4 \times 3,2 = 14,08 \text{ cm}^2$, and the scale factor for lengths, from the map to reality, is 25 000. Thus:

$$\mathcal{A}_{\text{actual}} = 14,08 \times 25\,000^2 = 8\,800\,000\,000 \text{ cm}^2.$$

Now $1 \text{ km}^2 = 10\,000\,000\,000 \text{ cm}^2$, so we again find $0,88 \text{ km}^2$.

Check. Both methods give the same result; if we had multiplied the area by only 25 000, they would have contradicted each other.

Writing model A2: calculating a length, an area or a volume in an enlargement or a reduction

The steps

① State the direction of the calculation, from the known figure to the target figure, and the scale factor for lengths in that direction: a scale of $\frac{1}{n}$ divides lengths by n from reality to the plan, and multiplies them by n from the plan to reality.

② Identify the dimension of the quantity: a length (or a perimeter) is multiplied by this factor, an area by its square, a volume by its cube; an angle measure is preserved.

③ Calculate in the given unit, then convert the result into the required unit.

④ Conclude with a sentence, including the unit, then check using the inverse calculation.

A scale of $\frac{1}{n}$ is a reduction: from reality to the plan or scale model, we divide by n ; in the other direction, we multiply by n .

On your paper

Scale model of a sailing boat at a scale of $\frac{1}{80}$.

a) The sailing boat is 18 m long. What is the length of the model, in cm?

b) The sail of the model has an area of 375 cm². What is the area of the actual sail, in m²?

c) The actual hull has a volume of 128 m³. What is the volume of the model's hull, in L?

a) ① From the sailing boat to the model, lengths are multiplied by $\frac{1}{80}$, that is, divided by 80.

② I am looking for a length: it is divided by 80.

③ $18 \div 80 = 0,225$ m, and $0,225 \text{ m} = 22,5 \text{ cm}$.

④ Conclusion: the model is 22,5 cm long.

Check: $0,225 \times 80 = 18$. ✓

b) ① From the model to the sailing boat, lengths are multiplied by 80.

② I am looking for an area: it is multiplied by $80^2 = 6\,400$.

③ $375 \times 6\,400 = 2\,400\,000 \text{ cm}^2$, and $2\,400\,000 \text{ cm}^2 = 240 \text{ m}^2$, because $1 \text{ m}^2 = 10\,000 \text{ cm}^2$.

④ Conclusion: the actual sail has an area of 240 m².

Check: $2\,400\,000 \div 6\,400 = 375$. ✓

c) ① From the sailing boat to the model, lengths are divided by 80.

② I am looking for a volume: it is divided by $80^3 = 512\,000$.

③ $128 \div 512\,000 = 0,000\,25 \text{ m}^3$, and $0,000\,25 \text{ m}^3 = 0,25 \text{ L}$, because $1 \text{ m}^3 = 1\,000 \text{ L}$.

④ Conclusion: the hull of the model has a volume of 0,25 L.

Check: $0,000\,25 \times 512\,000 = 128$. ✓

Your turn: a miniature tank (same model)

An actual tank with a volume of 2,56 m³ is reproduced at a scale of $\frac{1}{40}$. What is the volume of the miniature tank, in mL?

Answer: From the actual tank to the scale model, lengths are divided by 40; a volume is divided by $40^3 = 64\,000$; $2,56 \div 64\,000 = 0,000\,04 \text{ m}^3$, which is 0,04 L, or 40 mL. Check: $0,000\,04 \times 64\,000 = 2,56$. A small tank of a few tens of millilitres makes sense; several litres would point to an error in direction or forgetting to cube.

Quick questions (unaided)

1. On a plan at a scale of $\frac{1}{400}$, a surface measures 5,5 cm². What is its actual area in m²?

2. A scale model at a scale of $\frac{1}{25}$ represents an actual tank of 1,875 m³. What is the volume of the model in mL?

3. What question must you ask before choosing between k , k^2 and k^3 ?

Answers: 1) $5,5 \times 400^2 = 880\,000 \text{ cm}^2 = 88 \text{ m}^2$; 2) $1,875 \div 25^3 = 0,000\,12 \text{ m}^3 = 0,12 \text{ L} = 120 \text{ mL}$; 3) "is the required quantity a length, an area or a volume?"

→ Exercise sheet: exercises 2.4 e), 2.5 b), 3.1, 3.1 bis b), 3.2, 3.3, 3.4 b), 3.6, 3.8 d) and e), 4.1 c) and e), 4.2, 6.3, 6.5, 6.6, 7.1 and 7.2 (model A2)

4.3 Reverse direction: finding k from the areas

Square root of a positive number

For a positive number r , the square root of r is the positive number whose square is r . It is written $\sqrt{r}$.

It reads: $\sqrt{r}$ "square root of r ". In plain words: $\sqrt{r}$ is the positive number which, when multiplied by itself, gives r ; for example $\sqrt{144} = 12$, because $12^2 = 144$ and $12 > 0$.

Why look for the positive number whose square is the ratio of the areas?

In an enlargement or a reduction with scale factor k , areas are multiplied by k^2 (section 4.2): the ratio of the areas, target area divided by starting area, is therefore k^2 . Furthermore, $k > 0$ (it is a ratio of lengths), so k is the positive number whose square is this ratio: k is the square root of the ratio of the areas. We do not divide this ratio by 2: squaring is not doubling.

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Writing model A3: finding the scale factor k from the areas

The steps	On your paper
① Calculate the ratio of the areas, target area divided by starting area: it equals k^2 .	A figure with area 12,5 cm ² is enlarged into a figure with area 200 cm ² . What is the scale factor of the enlargement?
② Find the positive number whose square equals this ratio: this is k .	① The ratio of the areas is $\frac{200}{12,5} = 16$: this is k^2 .
③ Conclude, then check by multiplying the starting area by k^2 .	② The positive number whose square is 16 is 4, because $4^2 = 16$.
	③ The scale factor of the enlargement is $k = 4$.
	Check: $12,5 \times 4^2 = 12,5 \times 16 = 200$. ✓

→ Exercise sheet: exercises 3.1 d) and 3.3 b) (model A3)

Finding lengths from an area

Two similar triangles have respective areas of 32 cm² and 72 cm². The perimeter of the first is 28 cm and one of its altitudes is 5,6 cm. We want to find the corresponding quantities in the second triangle.

First we find k (model A3): the ratio of the areas is $\frac{72}{32} = \frac{9}{4} = 2,25$: this is k^2 . The positive number whose square is 2,25 is 1,5, because $1,5^2 = 2,25$: from the first triangle to the second, $k = 1,5$.

Next we calculate the lengths (model A2): a perimeter and an altitude are lengths; they are multiplied by k , not by k^2 .

$$P' = 28 \times 1,5 = 42 \text{ cm}, \quad h' = 5,6 \times 1,5 = 8,4 \text{ cm}.$$

Check. $\frac{42}{28} = \frac{8,4}{5,6} = 1,5$ and $\frac{72}{32} = 2,25 = 1,5^2$: the three ratios are consistent.

Your turn: reverse direction, to complete

A triangle T' is an enlargement of a triangle T . The area of T is 12 cm², and that of T' is 75 cm².

The ratio of the areas is $\frac{75}{12} = \dots\dots\dots$: this is k^2 .

The positive number whose square equals this ratio is $k = \dots\dots\dots$, because $(\dots\dots)^2 = \dots\dots$.

A side of T measures 8 cm; its corresponding side in T' therefore measures $\dots\dots\dots$.

Answers: $\frac{75}{12} = 6,25$; $k = 2,5$, because $2,5^2 = 6,25$; corresponding side $8 \times 2,5 = 20$ cm.

4.4 Checking a result

Sides and angles of a triangle (assumed)

In a triangle, the longest side is opposite the largest angle, and the shortest side is opposite the smallest angle.

In plain words: the wider an angle is, the longer the side opposite it; in triangle ABC from section 2.2 with angles 85°, 58° and 37°, the longest side is $[BC]$, opposite the 85° angle.

A single transformation, a single k

When two figures correspond under an enlargement or a reduction with scale factor k , all their corresponding quantities depend on the same k :

$$\frac{L'}{L} = \frac{P'}{P} = k, \quad \frac{A'}{A} = k^2, \quad \frac{V'}{V} = k^3.$$

It reads: $\frac{L'}{L}$ “ L prime over L ”: this is the quotient of the new length by the old length; P denotes a perimeter, $\mathcal{A}$ (“script A”) an area and V a volume. In plain words: a single number k must account for all the lengths, its square for all the areas and its cube for all the volumes; with $k = 10$, a length is multiplied by 10, an area by 100 and a volume by 1 000.

Why is a single mismatched value enough?

If the two figures truly correspond under a transformation with scale factor k , all quantities follow the rules above with the same k . We calculate k using one pair of corresponding lengths, then predict the other values. If a stated value differs from the predicted value, the data cannot all be accurate: a single contradiction is enough to reject the claim, whereas agreement only makes it consistent.

Writing model V3: testing a numerical claim using the scale factor k

The steps

- ① Write down the scale factor k for lengths, calculating it from two corresponding lengths if it is not given.
- ② Predict the expected value, using the factor for the dimension: k , k^2 or k^3 .
- ③ Compare the predicted value with the stated value.
- ④ Conclude in a sentence: if the values are equal, the claim is consistent; a single difference is enough to reject it. Then check the predicted value using the inverse calculation.

Here, Inès multiplied the volume by k instead of k^3 : $1,5 \times 0,4 = 0,6$.

On your paper

Two bottles have the same shape; the large one, 30 cm tall, holds 1,5 L; the small one is 12 cm tall. Inès claims: "The small bottle holds 0,6 L." Is she right?

- ① From the large bottle to the small one, lengths are multiplied by $k = \frac{12}{30} = 0,4$.
 - ② A volume is multiplied by $k^3 = 0,4^3 = 0,064$: the small bottle should hold $1,5 \times 0,064 = 0,096$ L.
 - ③ Inès states 0,6 L, and $0,096 \neq 0,6$.
 - ④ Conclusion: Inès is wrong; the small bottle holds 0,096 L, that is, 96 mL.
- Check: $0,096 \div 0,064 = 1,5$. ✓

→ Exercise sheet: exercises 3.5 b) and 3.6 b) (model V3)

Consistency check: do not accept all given values without checking

Two triangular signs are claimed to be similar. One side of the first measures 8,5 cm and its corresponding side on the second measures 11,9 cm. Their stated perimeters are 25 cm and 35 cm; their stated areas are 18 cm² and 34,2 cm².

From the first sign to the second, lengths are multiplied by $k = \frac{11,9}{8,5} = 1,4$.

A perimeter is multiplied by k : we expect $25 \times 1,4 = 35$ cm. The stated value is 35 cm: this value is consistent.

An area is multiplied by $k^2 = 1,4^2 = 1,96$: we expect $18 \times 1,96 = 35,28$ cm². The stated value is 34,2 cm², and $35,28 \neq 34,2$.

Conclusion: the four values cannot belong to the same pair of similar triangles. Even a convincing drawing would not fix this numerical contradiction.

Why is a rule enough, without any calculation?

The rules from the course notes hold for all shapes: if a statement contradicts one of them, it is false, with no need to calculate. To judge a result, we compare it with a familiar everyday quantity: an absurd order of magnitude points to an error in direction or dimension.

Writing model V1: showing without calculation that a statement is false, or that a result is plausible

The steps

- ① State the rule from the course notes that applies (dimension: k , k^2 or k^3 ; direction: $k > 1$ enlarges, $0 < k < 1$ reduces; in a triangle, the longest side is opposite the largest angle) or, to judge a result, a known reference quantity.
- ② Compare with the statement or result, and conclude: the statement is false, or the result is plausible.

One sentence per step: two short sentences are enough.

On your paper

a), b), c): each statement is false; explain why, without calculation. d): is the result plausible?

a) In an enlargement with scale factor 4, the area is multiplied by 8.

b) In a reduction, a 5 cm segment becomes a 15 cm segment.

c) In a triangle ABC with angles $\hat{A} = 42^\circ$, $\hat{B} = 63^\circ$ and $\hat{C} = 75^\circ$, the longest side is $[BC]$.

d) A $\frac{1}{80}$ scale model of an 18 m sailing boat measures 22,5 cm.

a) ① In an enlargement with scale factor 4, areas are multiplied by $4^2 = 16$.

② The area cannot therefore have been multiplied by 8: the statement is false.

b) ① In a reduction, the scale factor is strictly between 0 and 1: all lengths become shorter.

② A 5 cm segment cannot therefore become a longer 15 cm segment: the statement is false.

c) ① In a triangle, the longest side is opposite the largest angle.

② The largest angle, $\hat{C} = 75^\circ$, is opposite side $[AB]$; the longest side cannot therefore be $[BC]$: the statement is false.

d) ① A school ruler measures 20 cm.

② A 22,5 cm model, barely longer than a school ruler, fits on a shelf: for a model of a sailing boat, the result is plausible.

→ Exercise sheet: exercises 3.3 d), 4.1 e), 4.2 d) and 5.2 (model V1)

Quick questions (unaided)

1. A shape is enlarged with scale factor 11. By what factor is its area multiplied? Its perimeter?

2. A solid is reduced at a scale of $\frac{1}{10}$. By what factor is its volume divided?

3. The area of a shape has been multiplied by 49. What was the scale factor of the enlargement?

4. On a map at a scale of $\frac{1}{25\,000}$, what does 1 cm represent?

Answers: 1) area $\times 121$, perimeter $\times 11$ (the perimeter is a length!); 2) by $10^3 = 1\,000$; 3) $k = 7$, because $7^2 = 49$; 4) $25\,000\text{ cm} = 250\text{ m}$.

Classic mistakes: and why we make them

1. Confusing “similar” and “congruent”. Similar: same shape; congruent: same shape and same size ($k = 1$). The mistake comes from everyday language, where “similar” is often used to mean “the same”. (Exercise 1.5 of the exercise sheet.)

2. Taking the ratio between non-corresponding sides. The scale factor is read between sides that correspond through the angles: never between “the first side mentioned in each triangle”. This is the “match up” step, always done first. (Exercises 2.3 and 5.1.)

3. Multiplying the area by k instead of k^2 . An area is two-dimensional: each dimension is multiplied by k , so the area is multiplied by $k \times k$. (Exercises 3.1 and 5.2.)

4. Believing that a single pair of equal angles is enough. Two triangles can share an angle without being similar ($40^\circ\text{--}110^\circ\text{--}30^\circ$ versus $40^\circ\text{--}70^\circ\text{--}70^\circ$). Two pairs of equal angles are needed. (Exercise 1.1.)

5. Getting the direction of k wrong (multiplying when you should divide). Always write which triangle the ratio goes from and to: k is the final length divided by the corresponding initial length; then check consistency: the result must be larger if $k > 1$, smaller if $k < 1$. (Exercise 1.4.)

6. Concluding “by eye” from a drawing. A drawing can be misleading (and is not always to scale): justify using the angles or the three ratios. (Exercise 1.2.)

Programming two similar triangles

Same drawing program, modified lengths

To draw an equilateral triangle, repeat three times: move forward by the side length, then turn by 120° . The sprite turns by 120° on the outside of the triangle; the interior angle is $180^\circ - 120^\circ = 60^\circ$.

We store the length in a variable `side`. By replacing 40 with 100 without changing the turns or the number of repeats, we draw another equilateral triangle. The angles keep the same size; the triangles are similar, with scale factor $k = \frac{100}{40} = 2,5$ from the first to the second.

The perimeters are 120 steps and 300 steps: they are multiplied by 2,5. The area is multiplied by $2,5^2 = 6,25$. If the first area is $\mathcal{A}$, the second is $6,25\mathcal{A}$.

To run. Exercise 3.8 of the exercise sheet. Lift the pen up before moving the sprite between the two triangles, then restore the same starting direction. Moving with the pen down would add a segment that is not part of the triangles.

Predict before running

To go from a side of 40 steps to 10 steps, $k = \frac{10}{40} = 0,25$. The angles remain 60° , the perimeter is divided by 4, the area by $4^2 = 16$.

Running the program checks it; it is the equal angles and the ratio of the sides that justify similarity.

→ Exercise sheet: exercise 3.8 (models A1, R1 and A2)

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Summary sheet

The essentials of the chapter in one table		
	What you need to know	The reflex
Recognise	Similar triangles $\iff$ proportional sides $\iff$ pairs of equal angles (the symbol $\iff$ reads “if, and only if”). Two angles are enough (the sum of the three angles is 180°).	Given angles $\rightarrow$ calculate the 3 rd and compare (R1). Given sides $\rightarrow$ order them, then check all three ratios (R2).
Justify	Sources of equal angles: marked on the diagram, common angle, vertically opposite, alternate interior or corresponding (parallel lines), right angles.	Standard wording (D2): “We know that... / Similarly, we know that... / Now, if two triangles have two pairs of equal angles, then they are similar / Therefore triangles... are similar, with corresponding vertices...”.
Calculate	Match up (using the angles) / set up (table of corresponding sides) / solve (k , then multiply or divide) (L1).	Opposite the largest angle lies the longest side. Check: are the ratios equal? are the sides shorter if $k < 1$?
Enlarge / reduce	Scale factor k : lengths $\times k$, angles preserved, areas $\times k^2$, volumes $\times k^3$. Scale $\frac{1}{n}$: $k = \frac{1}{n}$ (A1, A2).	Is my quantity 1D, 2D or 3D? Reverse direction: the ratio of the areas is k^2 , then find $k > 0$ (A3).

→ Exercise sheet: for revision: brevet-style problems (French national exam at the end of Year 10) 6.1 to 6.6

→ Exercise sheet: optional 📖: 2.6, 3.7, 4.3 to 4.5, 5.4, 6.7 to 6.9 and 7.3 to 7.10

Flash questions (without help)
1. Angles 52° and 71° here, 57° and 71° there: similar? 2. $k = \frac{3}{7}$: effect on the perimeter? on the area? 3. Can the scale factor be negative? 4. An area goes from 20 cm^2 to 720 cm^2 (enlargement): what is k^2? and k? Answers. 1) yes, third angles 57° and 52°: same three angles; 2) perimeter $\times \frac{4}{3}$, area $\times \frac{16}{9}$; 3) no, $k > 0$ (it is a ratio of lengths); 4) $k^2 = \frac{720}{20} = 36$, so $k = 6$.

Self-assessment (multiple-choice): one correct answer per question

1. Two similar triangles have: A. the same sides B. proportional sides
C. the same perimeter

2. One triangle has angles of 40° and 70°; another has angles of 70° and 70°. They are necessarily: A. similar B. congruent C. neither

3. To prove that two triangles are similar, it is enough to have: A. one pair of equal angles B. two pairs of equal angles C. a common side

4. Triangles ABC and DEF are similar ($A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$). The side corresponding to $[BC]$ is: A. $[DE]$ B. $[DF]$ C. $[EF]$

5. Triangles ABC and DEF are similar, with $AB = 8$ and $DE = 20$ (corresponding sides). The scale factor from ABC to DEF is: A. 2,5 B. 0,4 C. 12
6. In the bowtie configuration, the angles at A (the point of intersection) are equal because they are: A. alternate interior B. vertically opposite C. corresponding

7. An enlargement with scale factor 8 multiplies the area by: A. 8 B. 16 C. 64

8. A reduction can have scale factor: A. 130 % B. 0,8 C. -2

9. A scale model at $\frac{1}{60}$: the actual volume is divided by: A. 60 B. 3 600 C. 216 000

10. The area of a shape increased from 2 cm² to 162 cm² in an enlargement. The scale factor k is: A. 81 B. 9 C. 160

Answers. 1 B; 2 A (third angles); 3 B; 4 C; 5 A; 6 B; 7 C; 8 B; 9 C (60°); 10 B ($k^2 = 81$).

Quick-reference index

Map of writing models			
Code	Type of exercise	The steps, in brief	Exercises on the sheet
R1	Similar? (using angles)	third angle, all three angles, conclusion	1.1, 3.8 d), 6.4 a)
R2	Similar? (using sides)	order, three ratios, conclusion	1.2
R3	Isosceles triangle: its angles	at the apex, at the base, one or two cases	1.1 e), 1.5 d) e)
D1	Two equal angles (parallel lines)	we know that, now, therefore	2.2 a), 6.1 a)
D2	Similar triangles (two angles)	one pair, another pair, now, therefore	1.3, 1.4 a), 2.2 b), 2.3 a) d), 2.4 c) d), 3.3 a), 3.4 a), 5.3 a), 6.1 b), 6.2 a)
D3	Similar triangles (one angle and two sides)	the angle, two ratios, now, therefore	2.5 a)
L1	Calculating lengths	match up, set up, k , calculate, check	1.4 b) c), 2.1 b) c), 2.2 c), 2.3, 2.4 c) d), 5.1 b), 5.3, 6.1 c), 6.2 b), 6.4 b) c)
A1	Scale factor k , scale	same unit, final over initial, enlargement or reduction	1.4 a), 2.1 a), 3.1 f) g), 3.1 bis a), 3.4 a), 3.5 a), 3.8 c) e), 4.1 b), 6.3 a)
A2	Length, area, volume	direction, dimension, calculation, conclusion and check	2.4 e), 2.5 b), 3.1, 3.1 bis b), 3.2, 3.3, 3.4 b), 3.6, 3.8 d) e), 4.1 c) e), 4.2, 6.3, 6.5, 6.6, 7.1, 7.2
A3	Finding k (areas)	ratio of areas, square root, check	3.1 d), 3.3 b)
V1	Without calculation	rule or reference, conclusion	3.3 d), 4.1 e), 4.2 d), 5.2
V2	Marking pupil work	read through, the mistake, the correction	5.1
V3	Testing a claim	k , expected value, comparison, conclusion	3.5 b), 3.6 b)
T2	True or false	answer, proof or counterexample	1.5
J1	Proving	we know that, now, therefore, ■	4.5 a) b)  , 5.4 a) 
J2	Right-angled triangle (Pythagoras)	longest side, two calculations, conclusion	2.4 a), 4.1 d)