



Worked solutions to the exercise sheet

Algebraic expressions: Notable identities

Difficulty: ★☆☆ very easy ★★☆☆ easy ★★★ medium ★★★★ hard

Track: 🏃 core track 🎯 recommended track 🏆 going further

The grey boxes 📝 show how to write out solutions on your paper.

To remember. The three notable identities (for all numbers a and b):

$$(a + b)^2 = a^2 + 2ab + b^2, \quad (a - b)^2 = a^2 - 2ab + b^2, \quad (a + b)(a - b) = a^2 - b^2.$$

Take a snapshot of the structure before factorising: common factor first; then two squares separated by $- \rightarrow (a + b)(a - b)$; three terms with square end terms and middle term equal to $2ab$ or $-2ab \rightarrow (a + b)^2$ or $(a - b)^2$.

Always check: re-expand mentally, or evaluate both expressions for $x = 1$ (agreement does not prove equality).

1. Nature of an expression and expanding

Quick warm-up (previous chapters) 🏃

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📝 a) $84 = 2^2 \times 3 \times 7$ and $126 = 2 \times 3^2 \times 7$, so $\frac{84}{126} = \frac{2^2 \times 3 \times 7}{2 \times 3^2 \times 7} = \boxed{\frac{2}{3}}$.

b) $2^3 \times 2^5 = 2^{3+5} = \boxed{2^8}$; $\frac{2^9}{2^4} = 2^{9-4} = \boxed{2^5}$.

📝 c) $0,0072 = \boxed{7,2 \times 10^{-3}}$.

d) $\frac{3}{4} - \frac{1}{6} = \frac{9}{12} - \frac{2}{12} = \boxed{\frac{7}{12}}$.

📝 e) The ratio of the areas is the square of the ratio of the sides: $2^2 = \boxed{4}$.

1.1. Sum or product? Sorting 🏃

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Method. Associated model: **L1**.

📝 a) In $3x + 2$, the last operation is adding $3x$ and 2 : it is a **sum**, with terms $3x$ and 2 .

b) In $5(x + 4)$, the last operation is multiplying 5 by $(x + 4)$: it is a **product**, with factors 5 and $(x + 4)$.

📝 c) In $(x + 1)(x - 3)$, the last operation is multiplying $(x + 1)$ by $(x - 3)$: it is a **product**, with factors $(x + 1)$ and $(x - 3)$.

d) In $x^2 - 9$, the last operation is subtracting 9 : it is a **sum**, with terms x^2 and -9 .

📝 e) In $7 - 5x$, the last operation is subtracting $5x$: it is a **sum**, with terms 7 and $-5x$.

f) In $(2x - 3)^2$, which reads $(2x - 3) \times (2x - 3)$, the last operation is multiplying $(2x - 3)$ by $(2x - 3)$: it is a **product**, with factors $(2x - 3)$ and $(2x - 3)$.

📝 g) In $3(x + 1) + 2$, the last operation is adding $3(x + 1)$ and 2 : it is a **sum**, with terms $3(x + 1)$ and 2 .

h) In $x(x + 1) + 3(x + 2)$, the last operation is adding $x(x + 1)$ and $3(x + 2)$: it is a **sum**, with terms $x(x + 1)$ and $3(x + 2)$.

📝 i) In $2x^2$, which reads $2 \times x \times x$, the last operation is a multiplication: it is a **product**, with factors 2 , x and x .

j) In $(x + 5)(x - 5)$, the last operation is multiplying $(x + 5)$ by $(x - 5)$: it is a **product**, with factors $(x + 5)$ and $(x - 5)$.

Warning! In f), a square is a product, even though its expansion, $4x^2 - 12x + 9$, is a sum: the factorised form and the expanded form of the same expression do not have the same nature.

1.2. Distributive property and opposite 🏃

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Method. Associated models: **D1** (question a)); **L2** (question b)); **L3** (question c)).

📝 a)

$$3(2x + 5) = 3 \times 2x + 3 \times 5$$

$$= \boxed{6x + 15}$$

Check ($x = 1$): $3 \times 7 = 21$ and $6 + 15 = 21$. ✓

$$-4x(6 - 3x) = -4x \times 6 + (-4x) \times (-3x)$$

$$= -24x + 12x^2$$

$$= \boxed{12x^2 - 24x}$$

Check ($x = 1$): $-4 \times 3 = -12$ and $12 - 24 = -12$. ✓



$$\begin{aligned}
7(3-x) + 2x &= 7 \times 3 + 7 \times (-x) + 2x \\
&= 21 - 7x + 2x \\
&= \boxed{-5x + 21}
\end{aligned}$$

calculate each product

collect like terms

Check ($x = 1$): $7 \times 2 + 2 = 16$ and $-5 + 21 = 16$. ✓



b) The opposite of $5 - 4x$ is written $-(5 - 4x)$. Change the sign of each of its terms: $-(5 - 4x) = \boxed{-5 + 4x}$.

Check: $(5 - 4x) + (-5 + 4x) = 0$. ✓

The opposite of $3x - 7$ is written $-(3x - 7)$. Change the sign of each of its terms: $-(3x - 7) = \boxed{-3x + 7}$.

Check: $(3x - 7) + (-3x + 7) = 0$. ✓



The opposite of $-2x + 9$ is written $-(-2x + 9)$. Change the sign of each of its terms: $-(-2x + 9) = \boxed{2x - 9}$.

Check: $(-2x + 9) + (2x - 9) = 0$. ✓

The opposite of $x^2 - 6x + 1$ is written $-(x^2 - 6x + 1)$. Change the sign of each of its terms: $-(x^2 - 6x + 1) = \boxed{-x^2 + 6x - 1}$.

Check: $(x^2 - 6x + 1) + (-x^2 + 6x - 1) = 0$. ✓



c) The like terms are $5x$ and $-2x$, then 3 and 8.

$$\begin{aligned}
5x + 3 - 2x + 8 &= (5 - 2)x + (3 + 8) \\
&= \boxed{3x + 11}
\end{aligned}$$

Check ($x = 1$): $5 + 3 - 2 + 8 = 14$ and $3 + 11 = 14$. ✓

The like terms are $4x^2$ and $3x^2$, then $-x$ and $7x$; the number -9 stands alone.

$$\begin{aligned}
4x^2 - x + 3x^2 + 7x - 9 &= (4 + 3)x^2 + (-1 + 7)x - 9 \\
&= \boxed{7x^2 + 6x - 9}
\end{aligned}$$

Check ($x = 1$): $4 - 1 + 3 + 7 - 9 = 4$ and $7 + 6 - 9 = 4$. ✓

1.3. Expanding double brackets: drills

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Method. Associated model: **D1**.



a)

$$\begin{aligned}
(x + 3)(x + 5) &= x \times x + x \times 5 + 3 \times x + 3 \times 5 \\
&= x^2 + 5x + 3x + 15 \\
&= \boxed{x^2 + 8x + 15}
\end{aligned}$$

Check ($x = 1$): $4 \times 6 = 24$ and $1 + 8 + 15 = 24$. ✓

b)

$$\begin{aligned}
(x - 3)(2x + 5) &= x \times 2x + x \times 5 + (-3) \times 2x + (-3) \times 5 \\
&= 2x^2 + 5x - 6x - 15 \\
&= \boxed{2x^2 - x - 15}
\end{aligned}$$

Check ($x = 1$): $(-2) \times 7 = -14$ and $2 - 1 - 15 = -14$. ✓



c)

$$\begin{aligned}
(3x + 1)(2x - 7) &= 3x \times 2x + 3x \times (-7) + 1 \times 2x + 1 \times (-7) \\
&= 6x^2 - 21x + 2x - 7 \\
&= \boxed{6x^2 - 19x - 7}
\end{aligned}$$

Check ($x = 1$): $4 \times (-5) = -20$ and $6 - 19 - 7 = -20$. ✓

d)

$$\begin{aligned}
(2x - 3)(5x + 7) &= 2x \times 5x + 2x \times 7 + (-3) \times 5x + (-3) \times 7 \\
&= 10x^2 + 14x - 15x - 21 \\
&= \boxed{10x^2 - x - 21}
\end{aligned}$$

Check ($x = 1$): $(-1) \times 12 = -12$ and $10 - 1 - 21 = -12$. ✓



e)

$$\begin{aligned}
-4x(6 - 3x) - 8x^2 &= -4x \times 6 + (-4x) \times (-3x) - 8x^2 \\
&= -24x + 12x^2 - 8x^2 \\
&= \boxed{4x^2 - 24x}
\end{aligned}$$

Check ($x = 1$): $-4 \times 3 - 8 = -20$ and $4 - 24 = -20$. ✓

f)

$$\begin{aligned}
3(2x + 1) - (6 - x) &= 3 \times 2x + 3 \times 1 - (6 - x) \\
&= 6x + 3 - 6 + x \\
&= \boxed{7x - 3}
\end{aligned}$$

Check ($x = 1$): $3 \times 3 - 5 = 4$ and $7 - 3 = 4$. ✓



g)

$$\begin{aligned}
(-3x + 5)(2x - 3) &= -3x \times 2x + (-3x) \times (-3) + 5 \times 2x + 5 \times (-3) \\
&= -6x^2 + 9x + 10x - 15 \\
&= \boxed{-6x^2 + 19x - 15}
\end{aligned}$$

Check ($x = 1$): $2 \times (-1) = -2$ and $-6 + 19 - 15 = -2$. ✓

h)

$$\begin{aligned}
(x + 6)(x - 2) - x^2 &= x \times x + x \times (-2) + 6 \times x + 6 \times (-2) - x^2 \\
&= x^2 - 2x + 6x - 12 - x^2 \\
&= \boxed{4x - 12}
\end{aligned}$$

Check ($x = 1$): $7 \times (-1) - 1 = -8$ and $4 - 12 = -8$. ✓

Warning! In f), writing $-(6 - x) = -6 - x$ amounts to changing the sign of only the first term in the brackets: we would get $6x + 3 - 6 - x = 5x - 3$, which equals 2 for $x = 1$ instead of 4. The check catches the error.

1.4. Mixed expressions to simplify

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Method. Associated model: **D1**.



a)

$$\begin{aligned}
A &= 5(4x - 7) + (x + 9)(5x - 6) \\
&= 5 \times 4x + 5 \times (-7) + x \times 5x + x \times (-6) + 9 \times 5x + 9 \times (-6) \\
&= 20x - 35 + 5x^2 - 6x + 45x - 54 \\
&= \boxed{5x^2 + 59x - 89}
\end{aligned}$$

I calculate each product

I collect like terms

Check ($x = 1$): $5 \times (-3) + 10 \times (-1) = -25$ and $5 + 59 - 89 = -25$. ✓

b)

$$\begin{aligned}
 B &= (x-5)(x+7) + (2x-4)(3x-1) \\
 &= x \times x + x \times 7 + (-5) \times x + (-5) \times 7 + 2x \times 3x + 2x \times (-1) + (-4) \times 3x + (-4) \times (-1) \\
 &= x^2 + 7x - 5x - 35 + 6x^2 - 2x - 12x + 4 \\
 &= \boxed{7x^2 - 12x - 31}
 \end{aligned}$$

I calculate each product

I collect like terms

Check ($x = 1$): $(-4) \times 8 + (-2) \times 2 = -36$ and $7 - 12 - 31 = -36$. ✓

c)

$$\begin{aligned}
 C &= 3x(x-3) - (x+7)(4x-2) \\
 &= 3x \times x + 3x \times (-3) - (x \times 4x + x \times (-2) + 7 \times 4x + 7 \times (-2)) \\
 &= 3x^2 - 9x - (4x^2 - 2x + 28x - 14) \\
 &= 3x^2 - 9x - 4x^2 + 2x - 28x + 14 \\
 &= \boxed{-x^2 - 35x + 14}
 \end{aligned}$$

I calculate each product

the $-$ sign changes the sign of each term in the brackets

I collect like terms

Check ($x = 1$): $3 \times (-2) - 8 \times 2 = -22$ and $-1 - 35 + 14 = -22$. ✓

d)

$$\begin{aligned}
 D &= (y-9)(3y+2) - (2y-7)(4y-5) \\
 &= y \times 3y + y \times 2 + (-9) \times 3y + (-9) \times 2 \\
 &\quad - (2y \times 4y + 2y \times (-5) + (-7) \times 4y + (-7) \times (-5)) \\
 &= 3y^2 + 2y - 27y - 18 - (8y^2 - 10y - 28y + 35) \\
 &= 3y^2 + 2y - 27y - 18 - 8y^2 + 10y + 28y - 35 \\
 &= \boxed{-5y^2 + 13y - 53}
 \end{aligned}$$

I calculate each product

the $-$ sign changes the sign of each term in the brackets

I collect like terms

Check ($y = 1$): $(-8) \times 5 - (-5) \times (-1) = -45$ and $-5 + 13 - 53 = -45$. ✓

1.5. Proving the double distributive property

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Method. Associated model: **J1**.

a) I expand the left-hand side. I let $k = a + b$.

$$\begin{aligned}
 (a+b)(c+d) &= k(c+d) && \text{I replace } a+b \text{ with } k \\
 &= \boxed{kc + kd} && \text{distributive property: } k(c+d) = kc + kd
 \end{aligned}$$

b) I replace k with $a + b$, then I expand each of the two products by distributing from the right.

$$\begin{aligned}
 kc + kd &= (a+b)c + (a+b)d && \text{I replace } k \text{ with } a+b \\
 &= \boxed{ac + bc + ad + bd} && \text{distributive property, twice: } (a+b)c = ac + bc \text{ and } (a+b)d = ad + bd
 \end{aligned}$$

c) I piece together the equalities obtained in a) and b), then I put the terms back in order.

$$\begin{aligned}
 (a+b)(c+d) &= ac + bc + ad + bd && \text{questions a) and b)} \\
 &= ac + ad + bc + bd && \text{addition is commutative: I swap } bc \text{ and } ad
 \end{aligned}$$

I obtain the right-hand side. Therefore, for all numbers a, b, c and d , $(a+b)(c+d) = ac + ad + bc + bd$. ■

I applied the distributive property $\boxed{3}$ times in all: once in a), twice in b).

d) I expand $(a+b)(c-d)$ along the same path. I let $k = a + b$.

$$\begin{aligned}
 (a+b)(c-d) &= k(c-d) && \text{I replace } a+b \text{ with } k \\
 &= kc - kd && \text{distributive property: } k(c-d) = kc - kd \\
 &= (a+b)c - (a+b)d && \text{I replace } k \text{ with } a+b \\
 &= ac + bc - (ad + bd) && \text{distributive property, twice: } (a+b)c = ac + bc \text{ and } (a+b)d = ad + bd \\
 &= ac + bc - ad - bd && \text{the } - \text{ sign changes the sign of each term in the brackets} \\
 &= ac - ad + bc - bd && \text{addition is commutative: I swap } bc \text{ and } -ad
 \end{aligned}$$

Therefore, for all numbers a, b, c and d , $\boxed{(a+b)(c-d) = ac - ad + bc - bd}$. ■

To remember. Giving a name to a block ($k = a + b$) makes it possible to treat it as a single number. The idea will be used again, for example in exercise 3.4, where a is the block $x + 1$.

2. Notable identities: expanding

2.1. Drills

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Method. Associated model: **D2**.

a) I use $(a+b)^2 = a^2 + 2ab + b^2$ with $a = x$ and $b = 3$.

$$(x+3)^2 = x^2 + 2 \times x \times 3 + 3^2$$

$$= \boxed{x^2 + 6x + 9}$$

Check ($x = 1$): $4^2 = 16$ and $1 + 6 + 9 = 16$. ✓

b) I use $(a+b)^2 = a^2 + 2ab + b^2$ with $a = x$ and $b = 10$.

$$(x+10)^2 = x^2 + 2 \times x \times 10 + 10^2$$

$$= \boxed{x^2 + 20x + 100}$$

Check ($x = 1$): $11^2 = 121$ and $1 + 20 + 100 = 121$. ✓

c) I use $(a-b)^2 = a^2 - 2ab + b^2$ with $a = x$ and $b = 4$.

$$(x-4)^2 = x^2 - 2 \times x \times 4 + 4^2$$

$$= \boxed{x^2 - 8x + 16}$$

Check ($x = 1$): $(-3)^2 = 9$ and $1 - 8 + 16 = 9$. ✓

d) I use $(a-b)^2 = a^2 - 2ab + b^2$ with $a = y$ and $b = 6$.

$$(y-6)^2 = y^2 - 2 \times y \times 6 + 6^2$$

$$= \boxed{y^2 - 12y + 36}$$

Check ($y = 1$): $(-5)^2 = 25$ and $1 - 12 + 36 = 25$. ✓

e) I use $(A-B)(A+B) = A^2 - B^2$ with $A = a$ and $B = 7$.

$$(a-7)(a+7) = a^2 - 7^2$$

$$= \boxed{a^2 - 49}$$

Check ($a = 1$): $(-6) \times 8 = -48$ and $1 - 49 = -48$. ✓

f) I use $(a+b)(a-b) = a^2 - b^2$ with $a = x$ and $b = 8$.

$$(x+8)(x-8) = x^2 - 8^2$$

$$= \boxed{x^2 - 64}$$

Check ($x = 1$): $9 \times (-7) = -63$ and $1 - 64 = -63$. ✓

g) I use $(a-b)(a+b) = a^2 - b^2$ with $a = t$ and $b = 5$.

$$(t-5)(t+5) = t^2 - 5^2$$

$$= \boxed{t^2 - 25}$$

Check ($t = 1$): $(-4) \times 6 = -24$ and $1 - 25 = -24$. ✓

h) I use $(a+b)^2 = a^2 + 2ab + b^2$ with $a = n$ and $b = 1$.

$$(n+1)^2 = n^2 + 2 \times n \times 1 + 1^2$$

$$= \boxed{n^2 + 2n + 1}$$

Check ($n = 1$): $2^2 = 4$ and $1 + 2 + 1 = 4$. ✓

i) I use $(A-B)^2 = A^2 - 2AB + B^2$ with $A = 4$ and $B = a$.

$$(4-a)^2 = 4^2 - 2 \times 4 \times a + a^2$$

$$= \boxed{16 - 8a + a^2}$$

Check ($a = 1$): $3^2 = 9$ and $16 - 8 + 1 = 9$. ✓

2.2. With coefficients

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Method. Associated model: **D2**.

a) I use $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 2x$ and $b = 1$.

$$(2x+1)^2 = (2x)^2 + 2 \times 2x \times 1 + 1^2$$

$$= \boxed{4x^2 + 4x + 1}$$

Check ($x = 1$): $3^2 = 9$ and $4 + 4 + 1 = 9$. ✓

b) I use $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 3x$ and $b = 7$.

$$(3x+7)^2 = (3x)^2 + 2 \times 3x \times 7 + 7^2$$

$$= \boxed{9x^2 + 42x + 49}$$

Check ($x = 1$): $10^2 = 100$ and $9 + 42 + 49 = 100$. ✓

c) I use $(a-b)^2 = a^2 - 2ab + b^2$ with $a = 3$ and $b = 2x$.

$$(3-2x)^2 = 3^2 - 2 \times 3 \times 2x + (2x)^2$$

$$= \boxed{9 - 12x + 4x^2}$$

Check ($x = 1$): $1^2 = 1$ and $9 - 12 + 4 = 1$. ✓

d) I use $(a-b)^2 = a^2 - 2ab + b^2$ with $a = 4x$ and $b = 3$.

$$(4x-3)^2 = (4x)^2 - 2 \times 4x \times 3 + 3^2$$

$$= \boxed{16x^2 - 24x + 9}$$

Check ($x = 1$): $1^2 = 1$ and $16 - 24 + 9 = 1$. ✓

e) I use $(a-b)(a+b) = a^2 - b^2$ with $a = 3x$ and $b = 2$.

$$(3x-2)(3x+2) = (3x)^2 - 2^2$$

$$= \boxed{9x^2 - 4}$$

Check ($x = 1$): $1 \times 5 = 5$ and $9 - 4 = 5$. ✓

f) I use $(a+b)(a-b) = a^2 - b^2$ with $a = 5x$ and $b = 7$.

$$(5x+7)(5x-7) = (5x)^2 - 7^2$$

$$= \boxed{25x^2 - 49}$$

Check ($x = 1$): $12 \times (-2) = -24$ and $25 - 49 = -24$. ✓

g) I use $(a-b)^2 = a^2 - 2ab + b^2$ with $a = y$ and $b = 2,5$.

$$(y-2,5)^2 = y^2 - 2 \times y \times 2,5 + 2,5^2$$

$$= \boxed{y^2 - 5y + 6,25}$$

Check ($y = 1$): $(-1,5)^2 = 2,25$ and $1 - 5 + 6,25 = 2,25$. ✓

h) I use $(A-B)(A+B) = A^2 - B^2$ with $A = 1$ and $B = a$.

$$(1-a)(1+a) = 1^2 - a^2$$

$$= \boxed{1 - a^2}$$

Check ($a = 1$): $0 \times 2 = 0$ and $1 - 1 = 0$. ✓

Warning! In c), writing $2x^2$ instead of $(2x)^2 = 4x^2$ would give $9 - 12x + 2x^2$, which equals -1 for $x = 1$ instead of 1 : the check catches the error.

2.3. The reverse direction: recognising an expanded form

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Method. Associated model: **F3**.

 a) Three terms, and the two outer ones are squares: it may be a perfect square. $x^2 = x^2$ and $25 = 5^2$, so $a = x$ and $b = 5$; the double product equals $2 \times x \times 5 = 10x$: that is indeed the middle term, preceded by a + sign.

$$\begin{aligned} x^2 + 10x + 25 &= x^2 + 2 \times x \times 5 + 5^2 \\ &= \boxed{(x + 5)^2} \quad a^2 + 2ab + b^2 = (a + b)^2 \end{aligned}$$

Check: $(x + 5)^2 = x^2 + 2 \times x \times 5 + 5^2 = x^2 + 10x + 25$. ✓

 Three terms, and the two outer ones are squares: it may be a perfect square. $x^2 = x^2$ and $9 = 3^2$, so $a = x$ and $b = 3$; the double product equals $2 \times x \times 3 = 6x$: that is indeed the middle term, preceded by a - sign.

$$\begin{aligned} x^2 - 6x + 9 &= x^2 - 2 \times x \times 3 + 3^2 \\ &= \boxed{(x - 3)^2} \quad a^2 - 2ab + b^2 = (a - b)^2 \end{aligned}$$

Check: $(x - 3)^2 = x^2 - 2 \times x \times 3 + 3^2 = x^2 - 6x + 9$. ✓

 Three terms, and the two outer ones are squares: it may be a perfect square. $4x^2 = (2x)^2$ and $9 = 3^2$, so $a = 2x$ and $b = 3$; the double product equals $2 \times 2x \times 3 = 12x$: that is indeed the middle term, preceded by a - sign.

$$\begin{aligned} 4x^2 - 12x + 9 &= (2x)^2 - 2 \times 2x \times 3 + 3^2 \\ &= \boxed{(2x - 3)^2} \quad a^2 - 2ab + b^2 = (a - b)^2 \end{aligned}$$

Check: $(2x - 3)^2 = (2x)^2 - 2 \times 2x \times 3 + 3^2 = 4x^2 - 12x + 9$. ✓

 Three terms, and the two outer ones are squares: it may be a perfect square. $4y^2 = (2y)^2$ and $100 = 10^2$, so $a = 2y$ and $b = 10$; the double product equals $2 \times 2y \times 10 = 40y$: that is indeed the middle term, preceded by a + sign.

$$\begin{aligned} 4y^2 + 40y + 100 &= (2y)^2 + 2 \times 2y \times 10 + 10^2 \\ &= \boxed{(2y + 10)^2} \quad a^2 + 2ab + b^2 = (a + b)^2 \end{aligned}$$

Check: $(2y + 10)^2 = (2y)^2 + 2 \times 2y \times 10 + 10^2 = 4y^2 + 40y + 100$. ✓

 Three terms, and the two outer ones are squares: it may be a perfect square. $49t^2 = (7t)^2$ and $1 = 1^2$, so $a = 7t$ and $b = 1$; the double product equals $2 \times 7t \times 1 = 14t$: that is indeed the middle term, preceded by a - sign.

$$\begin{aligned} 49t^2 - 14t + 1 &= (7t)^2 - 2 \times 7t \times 1 + 1^2 \\ &= \boxed{(7t - 1)^2} \quad a^2 - 2ab + b^2 = (a - b)^2 \end{aligned}$$

Check: $(7t - 1)^2 = (7t)^2 - 2 \times 7t \times 1 + 1^2 = 49t^2 - 14t + 1$. ✓

 b) $x^2 + \dots x + 9$: $x^2 = x^2$ and $9 = 3^2$, so $a = x$ and $b = 3$; the double product equals $2 \times x \times 3 = 6x$. The missing number can be $\boxed{6 \text{ or } -6}$: $x^2 + 6x + 9 = (x + 3)^2$ and $x^2 - 6x + 9 = (x - 3)^2$.

Check: $(x + 3)^2 = x^2 + 2 \times x \times 3 + 3^2 = x^2 + 6x + 9$ and $(x - 3)^2 = x^2 - 2 \times x \times 3 + 3^2 = x^2 - 6x + 9$. ✓

 $4x^2 - 4x + \dots$: $4x^2 = (2x)^2$, so $a = 2x$; the middle term $-4x$ is preceded by a - sign, and the double product $2 \times 2x \times b = 4x \times b$ must equal $4x$, so $b = 1$. The missing term is $1^2 = 1$: the missing number is $\boxed{1}$, and $4x^2 - 4x + 1 = (2x - 1)^2$.

Check: $(2x - 1)^2 = (2x)^2 - 2 \times 2x \times 1 + 1^2 = 4x^2 - 4x + 1$. ✓

 $\dots x^2 + 100x + 100$: $100 = 10^2$, so $b = 10$; the double product $2 \times a \times 10 = 20 \times a$ must equal $100x$, so $a = 5x$. The missing term is $(5x)^2 = 25x^2$: the missing number is $\boxed{25}$, and $25x^2 + 100x + 100 = (5x + 10)^2$.

Check: $(5x + 10)^2 = (5x)^2 + 2 \times 5x \times 10 + 10^2 = 25x^2 + 100x + 100$. ✓

 c) There are $\boxed{2}$ answers, 6 and -6 : $9 = 3^2 = (-3)^2$, and the double product $2 \times x \times 3$ can be preceded by a + sign or a - sign.

2.4. True or false?

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Method. Associated models: **T2**; **T3** (questions a), b), e) and f)); **J1** (question d)).

 a) FALSE. For $a = 1$, $(2a + 3)^2 = 5^2 = 25$ but $4a^2 + 9 = 4 + 9 = 13$: since $25 \neq 13$, the equality is not true for every number a .

 b) FALSE. For two numbers a and b , the statement is written $(a - b)^2 = a^2 - b^2$. For $a = 2$ and $b = 1$, $(a - b)^2 = 1^2 = 1$ but $a^2 - b^2 = 4 - 1 = 3$: since $1 \neq 3$, the equality is not true for all numbers a and b .

 c) TRUE. I calculate both sides separately: $26^2 - 25^2 = 676 - 625 = 51$ and $26 + 25 = 51$. Both sides are equal.

🔗 d) TRUE. I expand the left-hand side.

$$\begin{aligned}(a+5)^2 - (a-5)^2 &= a^2 + 10a + 25 - (a^2 - 10a + 25) & (A+B)^2 &= A^2 + 2AB + B^2 \text{ and } (A-B)^2 = A^2 - 2AB + B^2 \\ &= a^2 + 10a + 25 - a^2 + 10a - 25 & \text{the } - \text{ sign changes the sign of each term in the brackets} \\ &= 20a & \text{I collect like terms}\end{aligned}$$

I obtain the right-hand side. Therefore, for every number a , $(a+5)^2 - (a-5)^2 = 20a$. ■

🔗 e) FALSE. For $x = 1$, $x^2 - 12x + 9 = 1 - 12 + 9 = -2$: since a square is always greater than or equal to zero and $-2 < 0$, the number -2 is not the square of any number; the statement is not true for every number x .

🔗 f) FALSE. For $a = 1$ and $b = 1$, $(a+b)^2 = 2^2 = 4$ but $a^2 + b^2 = 1 + 1 = 2$: since $4 \neq 2$, the equality is not true for all numbers a and b .

🔗 g) TRUE. One example is enough: for $a = 1$ and $b = 0$, $(1+0)^2 = 1$ and $1^2 + 0^2 = 1$: both sides are equal for $a = 1$ and $b = 0$, so such numbers exist.

Warning! In a) and f), the false equality forgets the double product: $(2a+3)^2 = 4a^2 + 12a + 9$ and $(a+b)^2 = a^2 + 2ab + b^2$.

To remember. The statement “for all numbers a and b , $(a+b)^2 = a^2 + b^2$ ” is false (f), whereas this equality is true for some numbers (g): an identity must be true for all values of the letters.

2.5. With fractions 🏠

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Method. Associated model: **D2**.

🔗 a) I use $(a+b)(a-b) = a^2 - b^2$ with $a = \frac{1}{2}$ and $b = x$.

$$\begin{aligned}\left(\frac{1}{2} + x\right)\left(\frac{1}{2} - x\right) &= \left(\frac{1}{2}\right)^2 - x^2 \\ &= \boxed{\frac{1}{4} - x^2}\end{aligned}$$

Check ($x = 1$): $\frac{3}{2} \times \left(-\frac{1}{2}\right) = -\frac{3}{4}$ and $\frac{1}{4} - 1 = -\frac{3}{4}$. ✓

b) I use $(a-b)^2 = a^2 - 2ab + b^2$ with $a = \frac{x}{2}$ and $b = 3$.

$$\begin{aligned}\left(\frac{x}{2} - 3\right)^2 &= \left(\frac{x}{2}\right)^2 - 2 \times \frac{x}{2} \times 3 + 3^2 \\ &= \boxed{\frac{x^2}{4} - 3x + 9}\end{aligned}$$

Check ($x = 1$): $\left(-\frac{5}{2}\right)^2 = \frac{25}{4}$ and $\frac{1}{4} - 3 + 9 = \frac{25}{4}$. ✓

🔗 c) I use $(a-b)^2 = a^2 - 2ab + b^2$ with $a = \frac{1}{2}$ and $b = 3x$.

$$\begin{aligned}\left(\frac{1}{2} - 3x\right)^2 &= \left(\frac{1}{2}\right)^2 - 2 \times \frac{1}{2} \times 3x + (3x)^2 \\ &= \boxed{\frac{1}{4} - 3x + 9x^2}\end{aligned}$$

Check ($x = 1$): $\left(-\frac{5}{2}\right)^2 = \frac{25}{4}$ and $\frac{1}{4} - 3 + 9 = \frac{25}{4}$. ✓

d) I use $(a+b)^2 = a^2 + 2ab + b^2$ with $a = \frac{t}{3}$ and $b = 6$.

$$\begin{aligned}\left(\frac{t}{3} + 6\right)^2 &= \left(\frac{t}{3}\right)^2 + 2 \times \frac{t}{3} \times 6 + 6^2 \\ &= \boxed{\frac{t^2}{9} + 4t + 36}\end{aligned}$$

Check ($t = 1$): $\left(\frac{19}{3}\right)^2 = \frac{361}{9}$ and $\frac{1}{9} + 4 + 36 = \frac{361}{9}$. ✓

🔗 e) I use $(a-b)(a+b) = a^2 - b^2$ with $a = \frac{2x}{5}$ and $b = 1$.

$$\begin{aligned}\left(\frac{2x}{5} - 1\right)\left(\frac{2x}{5} + 1\right) &= \left(\frac{2x}{5}\right)^2 - 1^2 \\ &= \boxed{\frac{4x^2}{25} - 1}\end{aligned}$$

Check ($x = 1$): $\left(-\frac{3}{5}\right) \times \frac{7}{5} = -\frac{21}{25}$ and $\frac{4}{25} - 1 = -\frac{21}{25}$. ✓

To remember. To square a fraction, we square its numerator and its denominator: $\left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$.

2.6. Three forms, a single expression 🏠

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Method. Associated models: **V1** (questions a) and d)); **J3** (question b)); **J1** (question c)).

🔗 a) $A = a^2 - ab + b^2$. For $a = 5$ and $b = 3$:

$$\begin{aligned}A &= 5^2 - 5 \times 3 + 3^2 \\ &= 25 - 15 + 9 = 19\end{aligned}$$

For $a = 5$ and $b = 3$, $\boxed{A = 19}$.

$B = (a+b)^2 - 3ab$. For $a = 5$ and $b = 3$:

$$\begin{aligned}B &= (5+3)^2 - 3 \times 5 \times 3 \\ &= 64 - 45 = 19\end{aligned}$$

For $a = 5$ and $b = 3$, $\boxed{B = 19}$.

Both expressions have the same value.

🔗 b) No: the bet is not proved yet. The case $a = 5$, $b = 3$ does not prove the case $a = 2023$, $b = 1515$: we would have to recalculate everything... or prove the equality in general.

🔗 c) I expand B , for all numbers a and b .

$$\begin{aligned} B &= (a+b)^2 - 3ab \\ &= a^2 + 2ab + b^2 - 3ab \quad (a+b)^2 = a^2 + 2ab + b^2 \\ &= a^2 - ab + b^2 \quad \text{I collect like terms} \\ &= A \end{aligned}$$

I obtain A . Therefore, for all numbers a and b , $A = B$: Zoé's bet was won in advance, for 2023 and 1515 just as for any numbers, without writing down a single calculation. ■

🔗 d) $C = \frac{a^3 + b^3}{a + b}$. For $a = 5$ and $b = 3$:

$$\begin{aligned} C &= \frac{5^3 + 3^3}{5 + 3} \\ &= \frac{125 + 27}{8} = \frac{152}{8} = 19 \end{aligned}$$

For $a = 5$ and $b = 3$, $C = 19$: the same value as A and B . ✓

To remember. The fact accepted in d) comes from the identity $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, which can be checked by expanding the right-hand side using the double distributive property.

2.7. The formula for $(a+b+c)^2$ 🏠

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Method. The block trick once again: we name $(a+b)$ as a single number, apply a known identity, then expand the block again. This is the exact same lever $k = a+b$ as in the proof of the double distributive property (1.5). Associated models: **J3** (question a)); **J1** (questions b) and c)).

🔗 a) For $a = 1$, $b = 2$ and $c = 3$: $(1+2+3)^2 = 6^2 = 36$; and $1^2 + 2^2 + 3^2 + 2(1 \times 2 + 2 \times 3 + 3 \times 1) = 14 + 2 \times 11 = 36$. Both sides are equal. ✓

No, that is not enough: this test only establishes the case $a = 1$, $b = 2$, $c = 3$; it does not prove the formula for all numbers a , b and c .

🔗 b) I expand the left-hand side. I treat $(a+b)$ as a single block: I use $(A+B)^2 = A^2 + 2AB + B^2$ with $A = a+b$ and $B = c$.

$$\begin{aligned} (a+b+c)^2 &= [(a+b)+c]^2 & a+b+c &= (a+b)+c: \text{I group } a+b \text{ into a block} \\ &= (a+b)^2 + 2(a+b)c + c^2 & (A+B)^2 &= A^2 + 2AB + B^2 \end{aligned}$$

🔗 c) I complete the expansion and simplify.

$$\begin{aligned} (a+b)^2 + 2(a+b)c + c^2 &= a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \quad (a+b)^2 = a^2 + 2ab + b^2 \text{ and distributive property} \\ &= a^2 + b^2 + c^2 + 2(ab+bc+ca) \quad \text{I order the terms, then I factor out 2 from the double products} \end{aligned}$$

I obtain the right-hand side. Therefore, for all numbers a , b and c , $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca)$. ■

🔗 d) Three squares (a^2 , b^2 , c^2) and three double products ($2ab$, $2bc$, $2ca$), one for each pair of letters: a and b , b and c , c and a . Everything is there. ✓

3. Factorising

3.1. Common factor 🦋

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Method. Associated model: **F1**.

🔗 a) The common factor is 5: 5 is the highest common factor of 5 and 15, and no letter appears in both terms.

$$\begin{aligned} 5a + 15b &= \underline{5} \times a + \underline{5} \times 3b \\ &= \boxed{5(a+3b)} \end{aligned}$$

Check: $5(a+3b) = 5a + 15b$. ✓

b) The common factor is $3x$: 3 is the highest common factor of 12 and 15, and x appears in both terms, with the smallest exponent 1.

$$\begin{aligned} 12x^2 - 15x &= \underline{3x} \times 4x - \underline{3x} \times 5 \\ &= \boxed{3x(4x-5)} \end{aligned}$$

Check: $3x(4x-5) = 12x^2 - 15x$. ✓

🔗 c) The common factor is 7: 7 is the highest common factor of 7 and 35, and x does not appear in the term 35.

$$\begin{aligned} 7x - 35 &= \underline{7} \times x - \underline{7} \times 5 \\ &= \boxed{7(x-5)} \end{aligned}$$

Check: $7(x-5) = 7x - 35$. ✓

d) The common factor is x : 1 and 5 have no common divisor other than 1, and x appears in both terms, with the smallest exponent 1.

$$\begin{aligned} x^2 - 5x &= \underline{x} \times x - \underline{x} \times 5 \\ &= \boxed{x(x-5)} \end{aligned}$$

Check: $x(x-5) = x^2 - 5x$. ✓

e) The common factor is 2: 2 is the highest common factor of 8 and 2, and x does not appear in the term 8.

$$\begin{aligned} 8 + 2x &= 2 \times 4 + 2 \times x \\ &= \boxed{2(4 + x)} \end{aligned}$$

Check: $2(4 + x) = 8 + 2x$. ✓

f) The common factor is $6a$: 6 is the highest common factor of 30 and 36, and a appears in both terms, with the smallest exponent 1.

$$\begin{aligned} 30a + 36a^2 &= 6a \times 5 + 6a \times 6a \\ &= \boxed{6a(5 + 6a)} \end{aligned}$$

Check: $6a(5 + 6a) = 30a + 36a^2$. ✓

Warning! In b), $3(4x^2 - 5x)$ is already a product, but x remains common to both terms in the brackets. The problem asks to take out all visible common factors: we go all the way to $3x(4x - 5)$.

3.2. Compound common factor

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Method. Associated model: **F2**.

a) The bracket common to both terms is $(x - 2)$.

$$\begin{aligned} A &= (x - 2)(x + 3) + (x - 2)(x + 6) \\ &= (x - 2)[(x + 3) + (x + 6)] && \text{I factor out } (x - 2) \\ &= (x - 2)(x + 3 + x + 6) && \text{I remove the round brackets inside the square brackets} \\ &= \boxed{(x - 2)(2x + 9)} && \text{I simplify inside the square brackets} \end{aligned}$$

Check ($x = 1$): $(-1) \times 4 + (-1) \times 7 = -11$ and $(-1) \times 11 = -11$. ✓

b) The bracket common to both terms is $(x + 5)$.

$$\begin{aligned} B &= (x + 5)(4x - 2) - (x + 5)(9x + 1) \\ &= (x + 5)[(4x - 2) - (9x + 1)] && \text{I factor out } (x + 5) \\ &= (x + 5)(4x - 2 - 9x - 1) && \text{the } - \text{ sign changes the sign of each term in the brackets} \\ &= \boxed{(x + 5)(-5x - 3)} && \text{I simplify inside the square brackets} \end{aligned}$$

Check ($x = 1$): $6 \times 2 - 6 \times 10 = -48$ and $6 \times (-8) = -48$. ✓

c) The bracket common to both terms is $(3x + 8)$, because $(3x + 8)^2 = (3x + 8) \times (3x + 8)$.

$$\begin{aligned} C &= (3x + 8)^2 - (6x - 7)(3x + 8) \\ &= (3x + 8)[(3x + 8) - (6x - 7)] && \text{I factor out } (3x + 8) \\ &= (3x + 8)(3x + 8 - 6x + 7) && \text{the } - \text{ sign changes the sign of each term in the brackets} \\ &= (3x + 8)(-3x + 15) && \text{I simplify inside the square brackets} \\ &= \boxed{3(3x + 8)(-x + 5)} && \text{I factor out 3} \end{aligned}$$

Check ($x = 1$): $11^2 - (-1) \times 11 = 132$ and $3 \times 11 \times 4 = 132$. ✓

d) The bracket common to both terms is $(4x + 5)$, because $(4x + 5)^2 = (4x + 5) \times (4x + 5)$.

$$\begin{aligned} D &= (4x + 5)^2 - 5(4x + 5) \\ &= (4x + 5)[(4x + 5) - 5] && \text{I factor out } (4x + 5) \\ &= (4x + 5) \times 4x = \boxed{4x(4x + 5)} && \text{I simplify inside the square brackets} \end{aligned}$$

Check ($x = 1$): $9^2 - 5 \times 9 = 36$ and $4 \times 9 = 36$. ✓

Warning! In c), the common bracket is the second factor of the product $(6x - 7)(3x + 8)$: we look for it in each term, in any position.

3.3. Factorising with an identity: the pitfalls

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Method. Associated model: **F3**.

a) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $x^2 = x^2$ and $64 = 8^2$, so $a = x$ and $b = 8$.

$$\begin{aligned} x^2 - 64 &= x^2 - 8^2 \\ &= \boxed{(x + 8)(x - 8)} \end{aligned}$$

Check: $(x + 8)(x - 8) = x^2 - 8^2 = x^2 - 64$. ✓

b) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $49x^2 = (7x)^2$ and $16 = 4^2$, so $a = 7x$ and $b = 4$.

$$\begin{aligned} 49x^2 - 16 &= (7x)^2 - 4^2 \\ &= \boxed{(7x + 4)(7x - 4)} \end{aligned}$$

Check: $(7x + 4)(7x - 4) = (7x)^2 - 4^2 = 49x^2 - 16$. ✓

c) Three terms, and the two outer terms are squares: it may be a perfect square. $x^2 = x^2$ and $1 = 1^2$, so $a = x$ and $b = 1$; the double product is $2 \times x \times 1 = 2x$: this matches the middle term, preceded by a $+$ sign.

$$\begin{aligned} x^2 + 2x + 1 &= x^2 + 2 \times x \times 1 + 1^2 \\ &= \boxed{(x + 1)^2} \end{aligned}$$

Check: $(x + 1)^2 = x^2 + 2 \times x \times 1 + 1^2 = x^2 + 2x + 1$. ✓

d) Three terms, and the two outer terms are squares: it may be a perfect square. $9y^2 = (3y)^2$ and $16 = 4^2$, so $a = 3y$ and $b = 4$; the double product is $2 \times 3y \times 4 = 24y$: this matches the middle term, preceded by a $-$ sign.

$$\begin{aligned} 9y^2 - 24y + 16 &= (3y)^2 - 2 \times 3y \times 4 + 4^2 \\ &= \boxed{(3y - 4)^2} \end{aligned}$$

Check: $(3y - 4)^2 = (3y)^2 - 2 \times 3y \times 4 + 4^2 = 9y^2 - 24y + 16$. ✓

e) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $25 = 5^2$ and $100t^2 = (10t)^2$, so $a = 5$ and $b = 10t$.

$$\begin{aligned} 25 - 100t^2 &= 5^2 - (10t)^2 \\ &= (5 + 10t)(5 - 10t) \\ &= 5(1 + 2t) \times 5(1 - 2t) = \boxed{25(1 + 2t)(1 - 2t)} \end{aligned}$$

Check: $25(1 + 2t)(1 - 2t) = 25(1^2 - (2t)^2) = 25 - 100t^2$. ✓

f) Three terms, and the two outer terms are squares: it may be a perfect square. $64 = 8^2$ and $25x^2 = (5x)^2$, so $a = 8$ and $b = 5x$; the double product is $2 \times 8 \times 5x = 80x$: this matches the middle term, preceded by a $-$ sign.

$$\begin{aligned} 64 - 80x + 25x^2 &= 8^2 - 2 \times 8 \times 5x + (5x)^2 \\ &= \boxed{(8 - 5x)^2} \end{aligned}$$

Check: $(8 - 5x)^2 = 8^2 - 2 \times 8 \times 5x + (5x)^2 = 64 - 80x + 25x^2$. ✓

g) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $36z^2 = (6z)^2$ and $1 = 1^2$, so $a = 6z$ and $b = 1$.

$$\begin{aligned} 36z^2 - 1 &= (6z)^2 - 1^2 \\ &= \boxed{(6z + 1)(6z - 1)} \end{aligned}$$

Check: $(6z + 1)(6z - 1) = (6z)^2 - 1^2 = 36z^2 - 1$. ✓

h) Two terms separated by a $-$ sign; x^2 is a square, but 13 is not the square of an integer ($3^2 = 9$ and $4^2 = 16$), and no number known in this chapter has 13 as its square (we accept this): I cannot name b .

No factorisation for now: this stumbling block is temporary, as the positive number whose square is 13, written $\sqrt{13}$, will be defined in Chapter 02.

i) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $16x^2 = (4x)^2$ and $144 = 12^2$, so $a = 4x$ and $b = 12$.

$$\begin{aligned} 16x^2 - 144 &= (4x)^2 - 12^2 \\ &= (4x + 12)(4x - 12) \\ &= 4(x + 3) \times 4(x - 3) = \boxed{16(x + 3)(x - 3)} \end{aligned}$$

Check: $16(x + 3)(x - 3) = 16(x^2 - 3^2) = 16x^2 - 144$. ✓

j) Two terms separated by a $+$ sign: it is not a difference of two squares, and it is not a perfect square, which would have three terms.

None of the three identities applies: this stumbling block is not temporary within the scope of this chapter, because none of the three identities turns a sum of two squares into a product.

k) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $4x^2 = (2x)^2$ and $49 = 7^2$, so $a = 2x$ and $b = 7$.

$$\begin{aligned} 4x^2 - 49 &= (2x)^2 - 7^2 \\ &= \boxed{(2x + 7)(2x - 7)} \end{aligned}$$

Check: $(2x + 7)(2x - 7) = (2x)^2 - 7^2 = 4x^2 - 49$. ✓

l) Three terms, and the two outer terms are squares: it may be a perfect square. $x^2 = x^2$ and $100 = 10^2$, so $a = x$ and $b = 10$; the double product is $2 \times x \times 10 = 20x$: this matches the middle term, preceded by a $-$ sign.

$$\begin{aligned} x^2 - 20x + 100 &= x^2 - 2 \times x \times 10 + 10^2 \\ &= \boxed{(x - 10)^2} \end{aligned}$$

Check: $(x - 10)^2 = x^2 - 2 \times x \times 10 + 10^2 = x^2 - 20x + 100$. ✓

The two cases where none of the three identities allows us to conclude are h) and j).

To remember. In e) and i), we can also factor out the number first: $16x^2 - 144 = 16(x^2 - 9) = 16(x + 3)(x - 3)$. Both paths yield the same product; the second one reveals the factor 16 from the very first line.

3.4. Nested differences of squares

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Method. Associated model: **F3**.

a) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $(x + 1)^2$ is the square of the block $x + 1$ and $36 = 6^2$, so $a = x + 1$ and $b = 6$.

$$\begin{aligned} A &= (x + 1)^2 - 36 \\ &= (x + 1)^2 - 6^2 \\ &= [(x + 1) + 6][(x + 1) - 6] \quad a^2 - b^2 = (a + b)(a - b) \\ &= \boxed{(x + 7)(x - 5)} \quad \text{I simplify each factor} \end{aligned}$$

Check: $(x + 7)(x - 5) = x^2 - 5x + 7x - 35 = x^2 + 2x - 35$ and $(x + 1)^2 - 36 = x^2 + 2x + 1 - 36 = x^2 + 2x - 35$. ✓

b) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $25 = 5^2$ and $(2x + 3)^2$ is the square of the block $2x + 3$, so $a = 5$ and $b = 2x + 3$.

$$\begin{aligned} B &= 25 - (2x + 3)^2 \\ &= 5^2 - (2x + 3)^2 \\ &= [5 + (2x + 3)][5 - (2x + 3)] \quad a^2 - b^2 = (a + b)(a - b) \\ &= (5 + 2x + 3)(5 - 2x - 3) \quad \text{the } - \text{ sign changes the sign of each term inside the brackets} \\ &= (2x + 8)(-2x + 2) \quad \text{I simplify each factor} \\ &= 2(x + 4) \times 2(-x + 1) = \boxed{4(x + 4)(-x + 1)} \quad \text{I factor out 2 from each factor} \end{aligned}$$

Check: $4(x + 4)(-x + 1) = 4(-x^2 + x - 4x + 4) = -4x^2 - 12x + 16$ and $25 - (2x + 3)^2 = 25 - (4x^2 + 12x + 9) = -4x^2 - 12x + 16$. ✓

 c) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $4x^2 = (2x)^2$ and $(x-5)^2$ is the square of the block $x-5$, so $a = 2x$ and $b = x-5$.

$$\begin{aligned} C &= 4x^2 - (x-5)^2 \\ &= (2x)^2 - (x-5)^2 \\ &= [2x + (x-5)][2x - (x-5)] \quad a^2 - b^2 = (a+b)(a-b) \\ &= (2x + x-5)(2x - x+5) \quad \text{the } - \text{ sign changes the sign of each term inside the brackets} \\ &= \boxed{(3x-5)(x+5)} \quad \text{I simplify each factor} \end{aligned}$$

Check: $(3x-5)(x+5) = 3x^2 + 15x - 5x - 25 = 3x^2 + 10x - 25$ and $4x^2 - (x-5)^2 = 4x^2 - (x^2 - 10x + 25) = 3x^2 + 10x - 25$. ✓

 d) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $(8x-4)^2$ is the square of the block $8x-4$ and $9x^2 = (3x)^2$, so $a = 8x-4$ and $b = 3x$.

$$\begin{aligned} D &= (8x-4)^2 - 9x^2 \\ &= (8x-4)^2 - (3x)^2 \\ &= [(8x-4) + 3x][(8x-4) - 3x] \quad a^2 - b^2 = (a+b)(a-b) \\ &= \boxed{(11x-4)(5x-4)} \quad \text{I simplify each factor} \end{aligned}$$

Check: $(11x-4)(5x-4) = 55x^2 - 44x - 20x + 16 = 55x^2 - 64x + 16$ and $(8x-4)^2 - 9x^2 = 64x^2 - 64x + 16 - 9x^2 = 55x^2 - 64x + 16$. ✓

3.5. The two-way table

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Method. Associated models: **D2**; **F3**.

 I use $(a+b)^2 = a^2 + 2ab + b^2$ with $a = x$ and $b = 30$.

$$\begin{aligned} (x+30)^2 &= x^2 + 2 \times x \times 30 + 30^2 \\ &= \boxed{x^2 + 60x + 900} \end{aligned}$$

Check ($x = 1$): $31^2 = 961$ and $1 + 60 + 900 = 961$. ✓

Three terms, and the two outer terms are squares: it may be a perfect square. $x^2 = x^2$ and $36 = 6^2$, so $a = x$ and $b = 6$; the double product is $2 \times x \times 6 = 12x$: this matches the middle term, preceded by a $-$ sign.

$$\begin{aligned} x^2 - 12x + 36 &= x^2 - 2 \times x \times 6 + 6^2 \\ &= \boxed{(x-6)^2} \end{aligned}$$

Check: $(x-6)^2 = x^2 - 2 \times x \times 6 + 6^2 = x^2 - 12x + 36$. ✓

 Three terms, and the two outer terms are squares: it may be a perfect square. $25y^2 = (5y)^2$ and $25 = 5^2$, so $a = 5y$ and $b = 5$; the double product is $2 \times 5y \times 5 = 50y$: this matches the middle term, preceded by a $+$ sign.

$$\begin{aligned} 25y^2 + 50y + 25 &= (5y)^2 + 2 \times 5y \times 5 + 5^2 \\ &= \boxed{(5y+5)^2} \end{aligned}$$

Check: $(5y+5)^2 = (5y)^2 + 2 \times 5y \times 5 + 5^2 = 25y^2 + 50y + 25$. ✓

I use $(a-b)(a+b) = a^2 - b^2$ with $a = t$ and $b = 8$.

$$\begin{aligned} (t-8)(t+8) &= t^2 - 8^2 \\ &= \boxed{t^2 - 64} \end{aligned}$$

Check ($t = 1$): $(-7) \times 9 = -63$ and $1 - 64 = -63$. ✓

 Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $x^2 = x^2$ and $81 = 9^2$, so $a = x$ and $b = 9$.

$$\begin{aligned} x^2 - 81 &= x^2 - 9^2 \\ &= \boxed{(x+9)(x-9)} \end{aligned}$$

Check: $(x+9)(x-9) = x^2 - 9^2 = x^2 - 81$. ✓

I use $(a-b)^2 = a^2 - 2ab + b^2$ with $a = 7t$ and $b = 5$.

$$\begin{aligned} (7t-5)^2 &= (7t)^2 - 2 \times 7t \times 5 + 5^2 \\ &= \boxed{49t^2 - 70t + 25} \end{aligned}$$

Check ($t = 1$): $2^2 = 4$ and $49 - 70 + 25 = 4$. ✓

 The completed table:

Factorised form	Expanded form
$(x+30)^2$	$x^2 + 60x + 900$
$(x-6)^2$	$x^2 - 12x + 36$
$(5y+5)^2$	$25y^2 + 50y + 25$
$(t-8)(t+8)$	$t^2 - 64$
$(x+9)(x-9)$	$x^2 - 81$
$(7t-5)^2$	$49t^2 - 70t + 25$

To remember. In row 3, we can go further: $(5y+5)^2 = [5(y+1)]^2 = 25(y+1)^2$. Both expressions are exact factorised forms; the first shows the identity linking the two columns.

3.6. Louise's work (spot the mistake)

★★☆

Method. Associated models: **D1** (question a)); **T3** (question b)); **F3** (question c)).

 a)

$$\begin{aligned} 4x(x-1) &= 4x \times x + 4x \times (-1) \\ &= \boxed{4x^2 - 4x} \end{aligned} \quad \text{I calculate each product}$$

Check ($x=1$): $4 \times 1 \times 0 = 0$ and $4 - 4 = 0$. ✓

This is not E : the term $\boxed{+1}$ has disappeared.

 b) For $x=1$, $E = 4 \times 1^2 - 4 \times 1 + 1 = 1$ whereas $4x(x-1) = 4 \times 1 \times 0 = 0$: since $1 \neq 0$, the equality $E = 4x(x-1)$ is not true for all numbers x ; Louise's factorisation is wrong.

 c) Three terms, and the two outer terms are squares: it may be a perfect square. $4x^2 = (2x)^2$ and $1 = 1^2$, so $a = 2x$ and $b = 1$; the double product is $2 \times 2x \times 1 = 4x$: this matches the middle term, preceded by a $-$ sign.

$$\begin{aligned} E &= 4x^2 - 4x + 1 \\ &= (2x)^2 - 2 \times 2x \times 1 + 1^2 \\ &= \boxed{(2x-1)^2} \end{aligned} \quad a^2 - 2ab + b^2 = (a-b)^2$$

Check: $(2x-1)^2 = (2x)^2 - 2 \times 2x \times 1 + 1^2 = 4x^2 - 4x + 1$. ✓

 d) Testing with $x=1$ gives two different values: this is enough to disprove Louise's work. Re-expanding $(2x-1)^2$ gives back E for every number x : this proves the equality.

To remember. Agreement for a single value proves nothing: for $x=0$, $(x+2)^2$ and x^2+4 are both equal to 4, and yet $(x+2)^2 = x^2+4x+4$.

3.7. Factorising by grouping terms

★★★

Method. Four terms, neither a visible common factor nor an identity: group in pairs, factorise each group, and look for a common bracket. If the chosen grouping does not bring one out, try another: the failure of one grouping proves nothing. Associated models: **F1** (questions a) and b)); **F2** (questions a), c) and d)); **F3** (question c)).

 a) I group the terms in pairs and factorise each group: the bracket $(b+1)$ is common to both groups.

$$\begin{aligned} A &= ab + a + b + 1 \\ &= (ab + a) + (b + 1) \quad \text{I group the terms in pairs} \\ &= a(b + 1) + (b + 1) \times 1 \quad \text{I factorise each group} \\ &= \boxed{(a+1)(b+1)} \quad \text{I factor out } (b+1) \end{aligned}$$

Check: $(a+1)(b+1) = ab + a + b + 1$. ✓

 b)

$$\begin{aligned} B &= (a^2 - a) - (x^2 + x) \quad \text{I group the terms in pairs} \\ &= a(a-1) - x(x+1) \quad \text{I factorise each group} \end{aligned}$$

The brackets $(a-1)$ and $(x+1)$ are different, no common bracket appears, so I am stuck. This does not prove that the expression cannot be factorised: only this grouping fails, and I must try another grouping.

 c) I group the two squares together: the bracket $(a+x)$ becomes common to both groups.

$$\begin{aligned} B &= (a^2 - x^2) - (a + x) \quad \text{I group the two squares together} \\ &= (a-x)(a+x) - (a+x) \times 1 \quad A^2 - B^2 = (A-B)(A+B) \text{ with } A=a \text{ and } B=x \\ &= (a+x)[(a-x)-1] \quad \text{I factor out } (a+x) \\ &= \boxed{(a+x)(a-x-1)} \quad \text{I remove the inner brackets} \end{aligned}$$

 d) Check ($a=3$ and $x=1$): $B = 9 - 3 - 1 - 1 = 4$; and $(3+1)(3-1-1) = 4 \times 1 = 4$. ✓

4. Clever mental maths (the reward)

4.1. Squares near round numbers 🐦

★★☆☆

Method. Associated model: **C1**.

👉 a) I use $(a + b)^2 = a^2 + 2ab + b^2$ with $a = 100$ and $b = 1$, because $101 = 100 + 1$.

$$\begin{aligned} 101^2 &= (100 + 1)^2 = 100^2 + 2 \times 100 \times 1 + 1^2 \\ &= 10\,000 + 200 + 1 = \boxed{10\,201} \end{aligned}$$

Check: the units digit is that of $1 \times 1 = 1$, and 10 201 is slightly greater than $100^2 = 10\,000$. ✓

👉 c) I use $(a + b)(a - b) = a^2 - b^2$ with $a = 100$ and $b = 2$, because $102 = 100 + 2$ and $98 = 100 - 2$.

$$\begin{aligned} 102 \times 98 &= (100 + 2)(100 - 2) = 100^2 - 2^2 \\ &= 10\,000 - 4 = \boxed{9\,996} \end{aligned}$$

Check: the units digit is that of $2 \times 8 = 16$, namely 6, and 9996 is slightly smaller than $100^2 = 10\,000$. ✓

👉 e) I use $(a + b)^2 = a^2 + 2ab + b^2$ with $a = 20$ and $b = 1$, because $21 = 20 + 1$.

$$\begin{aligned} 21^2 &= (20 + 1)^2 = 20^2 + 2 \times 20 \times 1 + 1^2 \\ &= 400 + 40 + 1 = \boxed{441} \end{aligned}$$

Check: the units digit is that of $1 \times 1 = 1$, and 441 is slightly greater than $20^2 = 400$. ✓

👉 g) I use $(a + b)^2 = a^2 + 2ab + b^2$ with $a = 30$ and $b = 5$, because $35 = 30 + 5$.

$$\begin{aligned} 35^2 &= (30 + 5)^2 = 30^2 + 2 \times 30 \times 5 + 5^2 \\ &= 900 + 300 + 25 = \boxed{1\,225} \end{aligned}$$

Check: the units digit is that of $5 \times 5 = 25$, namely 5, and 1 225 lies between $30^2 = 900$ and $40^2 = 1\,600$. ✓

b) I use $(a - b)^2 = a^2 - 2ab + b^2$ with $a = 100$ and $b = 1$, because $99 = 100 - 1$.

$$\begin{aligned} 99^2 &= (100 - 1)^2 = 100^2 - 2 \times 100 \times 1 + 1^2 \\ &= 10\,000 - 200 + 1 = \boxed{9\,801} \end{aligned}$$

Check: the units digit is that of $9 \times 9 = 81$, namely 1, and 9 801 is slightly smaller than $100^2 = 10\,000$. ✓

d) I use $(a + b)(a - b) = a^2 - b^2$ with $a = 100$ and $b = 3$, because $103 = 100 + 3$ and $97 = 100 - 3$.

$$\begin{aligned} 103 \times 97 &= (100 + 3)(100 - 3) = 100^2 - 3^2 \\ &= 10\,000 - 9 = \boxed{9\,991} \end{aligned}$$

Check: the units digit is that of $3 \times 7 = 21$, namely 1, and 9991 is slightly smaller than $100^2 = 10\,000$. ✓

f) I use $(a - b)^2 = a^2 - 2ab + b^2$ with $a = 20$ and $b = 1$, because $19 = 20 - 1$.

$$\begin{aligned} 19^2 &= (20 - 1)^2 = 20^2 - 2 \times 20 \times 1 + 1^2 \\ &= 400 - 40 + 1 = \boxed{361} \end{aligned}$$

Check: the units digit is that of $9 \times 9 = 81$, namely 1, and 361 is slightly smaller than $20^2 = 400$. ✓

h) I use $(a + b)^2 = a^2 + 2ab + b^2$ with $a = 1\,000$ and $b = 1$, because $1\,001 = 1\,000 + 1$.

$$\begin{aligned} 1\,001^2 &= (1\,000 + 1)^2 = 1\,000^2 + 2 \times 1\,000 \times 1 + 1^2 \\ &= 1\,000\,000 + 2\,000 + 1 = \boxed{1\,002\,001} \end{aligned}$$

Check: the units digit is that of $1 \times 1 = 1$, and 1 002 001 is slightly greater than $1\,000^2 = 1\,000\,000$. ✓

4.2. In a single multiplication 🌀

★★☆☆

Method. Associated model: **C2**.

👉 a) I use $a^2 - b^2 = (a + b)(a - b)$ with $a = 73$ and $b = 67$.

$$\begin{aligned} 73^2 - 67^2 &= (73 + 67)(73 - 67) \\ &= 140 \times 6 = \boxed{840} \end{aligned}$$

Check: 73^2 ends like $3 \times 3 = 9$, in 9; 67^2 ends like $7 \times 7 = 49$, in 9; the difference therefore ends in 0, like 840. ✓

b) I use $a^2 - b^2 = (a + b)(a - b)$ with $a = 101$ and $b = 99$.

$$\begin{aligned} 101^2 - 99^2 &= (101 + 99)(101 - 99) \\ &= 200 \times 2 = \boxed{400} \end{aligned}$$

Check: 101^2 ends like $1 \times 1 = 1$, in 1; 99^2 ends like $9 \times 9 = 81$, in 1; the difference therefore ends in 0, like 400. ✓

👉 c) I use $a^2 - b^2 = (a + b)(a - b)$ with $a = 43$ and $b = 42$.

$$\begin{aligned} 43^2 - 42^2 &= (43 + 42)(43 - 42) \\ &= 85 \times 1 = \boxed{85} \end{aligned}$$

Check: 43^2 ends like $3 \times 3 = 9$, in 9; 42^2 ends like $2 \times 2 = 4$, in 4; the difference therefore ends in 5, like 85. ✓

d) I use $a^2 - b^2 = (a + b)(a - b)$ with $a = 8,5$ and $b = 1,5$.

$$\begin{aligned} 8,5^2 - 1,5^2 &= (8,5 + 1,5)(8,5 - 1,5) \\ &= 10 \times 7 = \boxed{70} \end{aligned}$$

Check: $8,5^2$ and $1,5^2$ both end in 25 after the decimal point, like $5 \times 5 = 25$; the difference therefore has no decimal places, like 70. ✓

👉 e) I use $a^2 + 2ab + b^2 = (a + b)^2$ with $a = 45$ and $b = 5$.

$$\begin{aligned} 45^2 + 2 \times 45 \times 5 + 5^2 &= (45 + 5)^2 \\ &= 50^2 = \boxed{2\,500} \end{aligned}$$

Check: the units digits of the three terms are 5, 0 and 5, and $5 + 0 + 5 = 10$ ends in 0, like 2 500. ✓

f) I use $a^2 - 2ab + b^2 = (a - b)^2$ with $a = 14$ and $b = 4$.

$$\begin{aligned} 14^2 - 2 \times 14 \times 4 + 4^2 &= (14 - 4)^2 \\ &= 10^2 = \boxed{100} \end{aligned}$$

Check: the units digits of the three terms are 6, 2 and 6, and $6 - 2 + 6 = 10$ ends in 0, like 100. ✓

4.3. The area between two squares 🌀

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Method. Associated models: **T1** (questions a) and b)); **C2** (question a)); **J1** (question c)).

👉 a) I know that the large square has side length 51 m and that the small square, placed in a corner of the large one, has side length 50 m. I am looking for the area $\mathcal{A}$ of the grey border: it is the area of the large square minus the area of the small square, so $\mathcal{A} = 51^2 - 50^2$, in m^2 . I use $a^2 - b^2 = (a + b)(a - b)$ with $a = 51$ and $b = 50$: $\mathcal{A} = (51 + 50)(51 - 50) = 101 \times 1 = 101$.

Conclusion: the grey border has an area of $\boxed{\mathcal{A} = 101 \text{ m}^2}$.

 b) Verification: the grey border is made up of two strips 1 m wide; each is about 50 m long, giving an area of about $2 \times 50 \times 1 = 100 \text{ m}^2$: 101 m^2 is plausible. ✓

 c) The area between the two squares is the area of the large square minus the area of the small square: I must prove that $(n+1)^2 - n^2 = 2n+1$. I expand the left-hand side.

$$\begin{aligned}(n+1)^2 - n^2 &= n^2 + 2n + 1 - n^2 & (a+b)^2 &= a^2 + 2ab + b^2 \\ &= 2n + 1 & & \text{I collect like terms}\end{aligned}$$

I obtain the right-hand side. Therefore, for every number $n > 0$, the area between the two squares is $(n+1)^2 - n^2 = 2n+1$. ■

In the course notes, the expression $2n+1$ is the form of an odd number when n is an integer; it also appears in the expansion $(n+1)^2 = n^2 + 2n + 1$ in Exercise 2.1 h): it is what must be added to n^2 to get $(n+1)^2$.

4.4. Champions: in the big league

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Method. Associated models: **C1** (questions a), b) and c)); **C2** (question d)).

 a) I use $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 1\,000$ and $b = 5$, because $1\,005 = 1\,000 + 5$.

$$\begin{aligned}1\,005^2 &= (1\,000 + 5)^2 = 1\,000^2 + 2 \times 1\,000 \times 5 + 5^2 \\ &= 1\,000\,000 + 10\,000 + 25 = 1\,010\,025\end{aligned}$$

Check: the units digit is that of $5 \times 5 = 25$, namely 5, and 1 010 025 is slightly greater than $1\,000^2 = 1\,000\,000$. ✓

 b) I use $(a+b)(a-b) = a^2 - b^2$ with $a = 10$ and $b = 0,1$, because $10,1 = 10 + 0,1$ and $9,9 = 10 - 0,1$.

$$\begin{aligned}10,1 \times 9,9 &= (10 + 0,1)(10 - 0,1) = 10^2 - 0,1^2 \\ &= 100 - 0,01 = 99,99\end{aligned}$$

Check: the last digit is that of $1 \times 9 = 9$, and 99,99 is slightly smaller than $10^2 = 100$. ✓

 c) I use the distributive property $k(a-b) = ka - kb$ with $k = 38$, $a = 1\,000$ and $b = 2$, because $998 = 1\,000 - 2$ and only 998 is close to a round number: an identity would require two factors equally distant from the same round number.

$$\begin{aligned}38 \times 998 &= 38 \times (1\,000 - 2) = 38 \times 1\,000 - 38 \times 2 \\ &= 38\,000 - 76 = 37\,924\end{aligned}$$

Check: the units digit is that of $8 \times 8 = 64$, namely 4, and 37 924 is slightly smaller than $38 \times 1\,000 = 38\,000$. ✓

 d) I use $a^2 - b^2 = (a+b)(a-b)$ with $a = 2\,026$ and $b = 2\,025$.

$$\begin{aligned}2\,026^2 - 2\,025^2 &= (2\,026 + 2\,025)(2\,026 - 2\,025) \\ &= 4\,051 \times 1 = 4\,051\end{aligned}$$

Check: $2\,026^2$ ends like $6 \times 6 = 36$, in 6; $2\,025^2$ ends like $5 \times 5 = 25$, in 5; the difference therefore ends in 1, like 4 051. ✓

So $2\,026^2 - 2\,025^2 - 4\,051 = 4\,051 - 4\,051 = 0$.

Warning! In c), an identity is possible, $38 \times 998 = (518 - 480)(518 + 480) = 518^2 - 480^2$, but it requires squaring two large numbers: it is applicable, not advantageous.

4.5. Checking without calculating

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Method. Associated models: **T3** (questions a), b) and d)); **C1** (question c)).

 a) For $x = 1$, $(x+10)^2 = 11^2 = 121$ whereas $x^2 + 100 = 1 + 100 = 101$: since $121 \neq 101$, the equality is not true for all numbers x .

 b) For $t = 1$, $25 - 100t^2 = 25 - 100 = -75$ whereas $(5 - 10t)^2 = (-5)^2 = 25$: since $-75 \neq 25$, the equality is not true for all numbers t .

 c) I use $(a-b)(a+b) = a^2 - b^2$ with $a = 50$ and $b = 3$, because $47 = 50 - 3$ and $53 = 50 + 3$: $47 \times 53 = (50 - 3)(50 + 3) = 50^2 - 3^2 = 2\,500 - 9 = 2\,491$, and since $2\,491 \neq 2\,500$, the statement is false.

 d) For $x = 1$, $(3x)^2 = 3^2 = 9$ whereas $3x^2 = 3 \times 1 = 3$: since $9 \neq 3$, the equality is not true for all numbers x .

To remember. Other one-sentence refutations are possible: in b), for $t = 1$, the left-hand side is strictly negative, whereas a square is always greater than or equal to zero; in c), 47×53 ends like $7 \times 3 = 21$, in 1, whereas 2 500 ends in 0.

4.6. Fixed perimeter, maximum area: the square wins

★★★☆

Method. Associated models: **D2** (question c)); **J1** (question d)).

 a) All three have perimeter $2 \times (12 + 8) = 2 \times (15 + 5) = 2 \times (10,5 + 9,5) = 40$ m. ✓ Areas, in m^2 :

$$12 \times 8 = \boxed{96}, \quad 15 \times 5 = \boxed{75}, \quad 10,5 \times 9,5 = \boxed{99,75}.$$

 b) If the perimeter is 40 m, the sum of the two dimensions is 20 m. Naming the larger one first, I write d for its difference from 10: the larger dimension is $10 + d$, with $d \geq 0$ because it is at least half of 20, and the smaller one is $20 - (10 + d) = 10 - d$. Since it is a length, $10 - d > 0$, so $d < 10$. Thus $0 \leq d < 10$.

 c) I use $(a + b)(a - b) = a^2 - b^2$ with $a = 10$ and $b = d$.

$$\begin{aligned} (10 + d)(10 - d) &= 10^2 - d^2 \\ &= 100 - d^2 \end{aligned}$$

Check ($d = 2$, sides 12 and 8): $100 - 2^2 = 96$, the area found in a). ✓

Since $d^2 \geq 0$, the area $100 - d^2$ never exceeds 100 m^2 ; it equals 100 m^2 only when $d = 0$, that is, for the square 10×10 .

 d) Let $x > 0$ and $y > 0$ be the sides, with $xy = 100$. I start from $(x - y)^2 \geq 0$, which is true because a square is always greater than or equal to zero.

$$\begin{aligned} x^2 - 2xy + y^2 &\geq 0 & (a - b)^2 &= a^2 - 2ab + b^2 \\ x^2 + 2xy + y^2 &\geq 4xy & \text{I add } 4xy &\text{ to both sides} \\ (x + y)^2 &\geq 4 \times 100 = 400 & a^2 + 2ab + b^2 &= (a + b)^2, \text{ and } xy = 100 \end{aligned}$$

Since $x + y > 0$, I get $x + y \geq 20$, so $2(x + y) \geq 40$. The equality $2(x + y) = 40$ requires $(x - y)^2 = 0$, hence $x = y = 10$.

Therefore, for every rectangle of area 100 m^2 , the perimeter is at least 40 m, and it equals 40 m only for the square 10×10 . ■

Conclusion: for a fixed perimeter, the square has the largest area; for a fixed area, it has the smallest perimeter.

To remember. The identity $a^2 - b^2$ solves an optimisation problem without any functions: among rectangles of a given perimeter, the square is the one with the largest area.

5. Calculation programs and proofs

5.1. Arthur's program: two trials are not enough to conclude for all x

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Method. Associated models: **P1** (questions a), b) and c)); **P2** (question c)); **J3** (question d)); **J1** (question d)).

 a) Starting with 2: $2 \times (-2) = -4$; $-4 + 5 = 1$; $1 \times 5 = 5$. I do get 5. ✓

 b) Starting with 3: $3 \times (-2) = -6$; $-6 + 5 = -1$; $-1 \times 5 = -5$. The result is -5 .

 c) Starting with x :

I multiply by -2 : $-2x$.

I add 5: $-2x + 5$.

I multiply the result by 5: $5(-2x + 5)$.

The result is $5(-2x + 5) = 5 \times (-2x) + 5 \times 5 = -10x + 25$.

I look for x such that $-10x + 25 = 0$.

$$\begin{aligned} -10x + 25 &= 0 \\ -10x &= 0 - 25 = -25 & \text{I subtract 25 from both sides} \\ x &= \frac{-25}{-10} = 2,5 & \text{I divide both sides by } -10 \end{aligned}$$

Conclusion: we must choose $x = 2,5$ as the starting number.

Verification: $2,5 \times (-2) = -5$; $-5 + 5 = 0$; $0 \times 5 = 0$. ✓

 d) For $x = 2$, $(x - 5)^2 - x^2 = (2 - 5)^2 - 2^2 = (-3)^2 - 4 = 9 - 4 = 5$; for $x = 3$, $(x - 5)^2 - x^2 = (3 - 5)^2 - 3^2 = (-2)^2 - 9 = 4 - 9 = -5$: these are the results of the program found in a) and b).

These examples only establish the tested cases: the statement concerns infinitely many numbers, so a proof using a letter is required. These two tests are not enough to decide.

I expand each expression.

$$\begin{aligned} 5(-2x + 5) &= -10x + 25 & \text{distributive property} \\ (x - 5)^2 - x^2 &= x^2 - 10x + 25 - x^2 & (a - b)^2 = a^2 - 2ab + b^2 \\ &= -10x + 25 & \text{I collect like terms} \end{aligned}$$

Both expressions are equal to $-10x + 25$. Therefore, for every number x , $(x - 5)^2 - x^2$ gives the result of the program: Arthur is right. ■

 Modify, create, solve

1. I predict that the starting number that gives 0 does not change: starting with 2,5, the result before the last multiplication is already 0, as in question c), and $0 \times 3 = 0$. I check this algebraically.

Starting with x :

I multiply by -2 : $-2x$.

I add 5: $-2x + 5$.

I multiply the result by 3: $3(-2x + 5)$.

The result is $3(-2x + 5) = 3 \times (-2x) + 3 \times 5 = -6x + 15$.

I look for x such that $-6x + 15 = 0$.

$$-6x + 15 = 0$$

$$-6x = 0 - 15 = -15 \quad \text{I subtract 15 from both sides}$$

$$x = \frac{-15}{-6} = 2,5 \quad \text{I divide both sides by } -6$$

Conclusion: we must still choose $x = 2,5$ as the starting number; it does not change.

Check: $2,5 \times (-2) = -5$; $-5 + 5 = 0$; $0 \times 3 = 0$. ✓

 2. I suggest this calculation program: "Choose a number; multiply it by 3; subtract 6; multiply the result by 2." For a starting number x , this program gives $2(3x - 6) = 2 \times 3x + 2 \times (-6) = \boxed{6x - 12}$.

3. Starting with x :

I multiply by 3: $3x$.

I subtract 6: $3x - 6$.

I multiply the result by 2: $2(3x - 6)$.

The result is $2(3x - 6) = 2 \times 3x + 2 \times (-6) = \boxed{6x - 12}$.

Starting with 0: $0 \times 3 = 0$; $0 - 6 = -6$; $-6 \times 2 = -12$. I do indeed get $6 \times 0 - 12 = -12$. ✓

Starting with 5: $5 \times 3 = 15$; $15 - 6 = 9$; $9 \times 2 = 18$. I do indeed get $6 \times 5 - 12 = 30 - 12 = 18$. ✓

Warning! A common mistake in d) is to expand $(x - 5)^2$ as $x^2 - 25$: the cross-term $-10x$ disappears, and Arthur's expression would be equal to -25 for every number x , whereas the program gives 5 for $x = 2$.

To remember. If one of the two tests had not matched, it would have been enough to prove Arthur wrong: it is a counterexample.

5.2. Two programs, the same result?

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Method. Related models: **P1** (questions a) and b)); **J3** (question a)); **J1** (question b)).

 a) Starting with 3, program A: $3 - 1 = 2$; $2^2 = 4$; $4 - 1 = 3$. The result is $\boxed{3}$.

Starting with 3, program B: $3 - 2 = 1$; $1 \times 3 = 3$. The result is $\boxed{3}$.

Starting with 10, program A: $10 - 1 = 9$; $9^2 = 81$; $81 - 1 = 80$. The result is $\boxed{80}$.

Starting with 10, program B: $10 - 2 = 8$; $8 \times 10 = 80$. The result is $\boxed{80}$.

Starting with -5 , program A: $-5 - 1 = -6$; $(-6)^2 = 36$; $36 - 1 = 35$. The result is $\boxed{35}$.

Starting with -5 , program B: $-5 - 2 = -7$; $-7 \times (-5) = 35$. The result is $\boxed{35}$.

For each of the three numbers, both programs give the same result. It seems that both programs always give the same result.

These examples only establish the cases tested: the statement is about infinitely many numbers, so a proof using a letter is needed.

 b) Starting with x , program A:

I subtract 1: $x - 1$.

I square the result: $(x - 1)^2$.

I subtract 1: $(x - 1)^2 - 1$.

Starting with x , program B:

I subtract 2: $x - 2$.

I multiply by the chosen number: $(x - 2) \times x = x(x - 2)$.

I expand each expression.

$$(x - 1)^2 - 1 = x^2 - 2x + 1 - 1 = x^2 - 2x \quad (a - b)^2 = a^2 - 2ab + b^2$$

$$x(x - 2) = x^2 - 2x$$

distributive property

Both expressions are equal to $x^2 - 2x$. Therefore, for any number x , programs A and B give the same result: the conjecture is proved. ■

Warning! Starting with -5 , program A squares -6 : $(-6)^2 = 36$, and not -36 . The square of a negative number is positive.

5.3. Brevet-style: the squared program

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Method. Related models: **P1** (questions a), b) and c)); **P2** (question d)).

 a) Starting with 1: $1 + 1 = 2$; $2^2 = 4$; $4 - 1^2 = 4 - 1 = 3$. I do indeed get 3. ✓

 b) Starting with 2: $2 + 1 = 3$; $3^2 = 9$; $9 - 2^2 = 9 - 4 = 5$. The result is $\boxed{5}$.

 c) Starting with x :

I add 1: $x + 1$.

I calculate the square of the result: $(x + 1)^2$.

I subtract the square of the starting number: $(x + 1)^2 - x^2$.

The result is $P = (x + 1)^2 - x^2$. I expand it by expanding double brackets:

$$\begin{aligned} P &= (x + 1)(x + 1) - x^2 & (x + 1)^2 &= (x + 1) \times (x + 1) \\ &= x^2 + x + x + 1 - x^2 & \text{expanding double brackets: four products} \\ &= \boxed{2x + 1} & \text{I collect like terms} \end{aligned}$$

 d) I look for x such that $2x + 1 = 15$.

$$2x + 1 = 15$$

$$2x = 15 - 1 = 14 \quad \text{I subtract 1 from both sides}$$

$$x = \frac{14}{2} = 7 \quad \text{I divide both sides by 2}$$

Conclusion: we must choose $\boxed{x = 7}$ as the starting number.

Check: $7 + 1 = 8$; $8^2 = 64$; $64 - 7^2 = 64 - 49 = 15$. ✓

To remember. After studying factorisation, the difference of two squares, with $a = x + 1$ and $b = x$, gives the same result without expanding a square: $P = [(x + 1) + x][(x + 1) - x] = (2x + 1) \times 1 = 2x + 1$.

5.4. Spreadsheet conjecture, algebraic proof: three consecutive integers

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Method. Related models: **J3** (questions a) and c)); **J2** (questions d) and e)); **T3** (question f)).

 a) $(-8) + (-7) + (-6) = -21$ and $-21 \div 3 = -7$; $7 + 8 + 9 = 24$ and $24 \div 3 = 8$; $199 + 200 + 201 = 600$ and $600 \div 3 = 200$.

Each sum is divisible by 3, and each time the quotient is the middle integer.

 b) After extending the column, column A contains the 200 integers from -100 to 99 . In B2, I enter $\boxed{=A2+1}$: the second integer follows the first. In C2, I enter $\boxed{=A2+2}$. In D2, I enter $\boxed{=A2+B2+C2}$: the sum of the three integers. I then fill B2:D2 down to row 201.

 c) In all 200 rows displayed, the sum is divisible by 3, and the quotient is the integer in column B. It seems that the sum of three consecutive integers is always a multiple of 3, and in fact three times the middle integer.

These examples only establish the cases tested, the 200 cases where the first integer goes from -100 to 99 : the statement is about infinitely many integers, so a proof using a letter is needed.

 d) Let n be any integer. Three consecutive integers can be written as n , $n + 1$ and $n + 2$: the letter n denotes the first one. I calculate their sum:

$$\begin{aligned} n + (n + 1) + (n + 2) &= 3n + 3 & \text{I collect like terms} \\ &= 3(n + 1) & \text{I factor out 3} \end{aligned}$$

Now n is an integer, so $n + 1$ is an integer: $3(n + 1)$ is a multiple of 3.

Therefore, for every integer n , the sum of three consecutive integers is a multiple of 3. ■

 e) From question d), the sum is equal to $3(n + 1)$. Now $n + 1$ is the second of the three integers.

Therefore, for every integer n , the sum of three consecutive integers is three times $\boxed{\text{the second integer, the middle one}}$. ■

f) For the consecutive integers 1, 2, 3 and 4, the sum is $1 + 2 + 3 + 4 = 10$, and 10 is not a multiple of 4 ($4 \times 2 = 8$ and $4 \times 3 = 12$): the statement “the sum of four consecutive integers is a multiple of 4” is not true for all integers. $\boxed{\text{No}}$.

To remember. Algebra explains the counterexample in f): $n + (n + 1) + (n + 2) + (n + 3) = 4n + 6 = 4(n + 1) + 2$. The remainder when dividing by 4 is always 2: the sum of four consecutive integers is never a multiple of 4.

5.5. The difference of the squares of two consecutive odd numbers

★★★★

Method. Related models: **J3** (question a)); **J2** (question b)).

 a) $5^2 - 3^2 = 16 = 8 \times 2$; $9^2 - 7^2 = 32 = 8 \times 4$; $13^2 - 11^2 = 48 = 8 \times 6$.

These three differences are multiples of 8. ✓

 b) Let n be any integer. Two consecutive odd numbers can be written as $2n + 1$ and $2n + 3$.

I calculate the difference of their squares using the factorisation $a^2 - b^2 = (a + b)(a - b)$, with $a = 2n + 3$ and $b = 2n + 1$:

$$\begin{aligned} (2n + 3)^2 - (2n + 1)^2 &= [(2n + 3) + (2n + 1)][(2n + 3) - (2n + 1)] & a^2 - b^2 &= (a + b)(a - b) \\ &= (4n + 4) \times 2 & \text{I simplify each factor} \\ &= 8n + 8 & \text{distributive property} \\ &= 8(n + 1) & \text{I factor out 8} \end{aligned}$$

Now n is an integer, so $n + 1$ is an integer: $8(n + 1)$ is a multiple of 8.

Therefore, for every integer n , the difference of the squares of two consecutive odd numbers is a $\boxed{\text{multiple of 8}}$: Anabelle’s statement is true. ■

🔗 c) In $(4n + 4) \times 2$: the factor $4n + 4$ is the sum of the two odd numbers, and the factor 2 is the gap between them: the difference of two squares can be written as “sum $\times$ gap”.

To remember. Another proof of b): expanding with $(a + b)^2 = a^2 + 2ab + b^2$ gives $4n^2 + 12n + 9 - (4n^2 + 4n + 1) = 8n + 8$. Now $8n$ and 8 are multiples of 8; so their sum is as well, and $8n + 8 = 8(n + 1)$ makes the quotient explicit.

5.6. Divisible by 7 (echo of Chapter 01) 🧩

★★★★

Method. Related model: **J2**.

🔗 Let n be any positive integer.

The common factor of the three terms is 2^n (the smallest power):

$$\begin{aligned} 2^n + 2^{n+1} + 2^{n+2} &= 2^n \times 1 + 2^n \times 2 + 2^n \times 4 & 2^{n+1} &= 2^n \times 2 \\ & & \text{and } 2^{n+2} &= 2^n \times 4 \\ &= 2^n(1 + 2 + 4) & \text{I factor out } 2^n \\ &= 7 \times 2^n & \text{I calculate the bracket} \end{aligned}$$

Now n is a positive integer, so 2^n is an integer: 7×2^n is the product of 7 and an integer.

Therefore, for every positive integer n , $2^n + 2^{n+1} + 2^{n+2}$ is divisible by 7. ■

To remember. This is the watchword from Chapter 01, promoted to a proof tool: a sum of powers cannot be “collected”, it must be factorised ($4^{15} + 8^{10} = 2^{30} + 2^{30} = 2 \times 2^{30} = 2^{31}$, remember...).

5.7. The checked tablecloth 🧩

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Method. Count the same collection in two ways (by rows, then by types), translate each count into algebra, and check that the sum gives the total back: this is double counting, just like the mosaic square in the course notes (Part 1). Related models: **D2** and **J1** (question d)).

🔗 a) In the figure ($n = 2$): 9 red, 4 white, 12 striped. Total: $9 + 4 + 12 = 25 = 5^2$. ✓

🔗 b) The rows with red corners are rows $n^{\text{os}} 1, 3, 5, \dots, 2n + 1$: there are $n + 1$ of them, and each contains $n + 1$ red and n striped squares. The other rows ($n^{\text{os}} 2, 4, \dots, 2n$) are n in number, and each contains $n + 1$ striped and n white squares.

🔗 c) Red: $(n + 1) \times (n + 1) = \boxed{(n + 1)^2}$; white: $n \times n = \boxed{n^2}$; striped: $(n + 1) \times n + n \times (n + 1) = \boxed{2n(n + 1)}$.

🔗 d) I expand and simplify the sum of the three expressions.

$$\begin{aligned} (n + 1)^2 + n^2 + 2n(n + 1) &= n^2 + 2n + 1 + n^2 + 2n^2 + 2n & (a + b)^2 &= a^2 + 2ab + b^2 \text{ and distributive property} \\ &= 4n^2 + 4n + 1 & \text{I collect like terms} \\ &= \boxed{(2n + 1)^2} & a^2 + 2ab + b^2 &= (a + b)^2 \text{ with } a = 2n \text{ and } b = 1 \end{aligned}$$

I get $(2n + 1)^2$ back. ✓

This check was predictable: the red, white and striped squares fill the entire square of side $2n + 1$, which contains $(2n + 1)^2$ squares. In fact, I can read the equality backwards: $(2n + 1)^2 = [(n + 1) + n]^2 = (n + 1)^2 + 2n(n + 1) + n^2$, which is the square of a sum with $a = n + 1$ and $b = n$.

Conclusion: for every integer $n \geq 1$, the tablecloth of side $2n + 1$ contains $(n + 1)^2$ red squares, n^2 white squares and $2n(n + 1)$ striped squares, and the sum does indeed give $(2n + 1)^2$ back.

🔗 e) Check for $n = 2$: $(n + 1)^2 = 9$, $n^2 = 4$ and $2n(n + 1) = 12$: I get the counts from a) back. ✓

5.8. A percentage decrease followed by the same percentage increase 🧩

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Method. Related models: **D2** and **J1** (question c)).

🔗 a) $1000 \xrightarrow{-50\%} 500 \xrightarrow{+50\%} 500 + 250 = 750$ €. The portfolio has lost 250 €, that is 25 % of the initial capital: not 0 %!

🔗 b) A 10 % loss followed by a 10 % gain: $0,9 \times 1,1 = 0,99$, the capital is multiplied by 0,99 (a 1 % loss). With 30 %: $0,7 \times 1,3 = 0,91$, the capital is multiplied by 0,91 (a 9 % loss). The number by which the capital is multiplied always remains slightly less than 1.

🔗 c) Losing a proportion x multiplies by $(1 - x)$; regaining a proportion x multiplies by $(1 + x)$. The overall multiplier is therefore $(1 - x)(1 + x)$. I use $(a - b)(a + b) = a^2 - b^2$ with $a = 1$ and $b = x$.

$$\begin{aligned} (1 - x)(1 + x) &= 1^2 - x^2 \\ &= 1 - x^2 \end{aligned}$$

Check ($x = 0,5$): $1 - 0,5^2 = 0,75$, and $0,75 \times 1000 = 750$ €, the result from a). ✓

Now $0 < x < 1$, so $x \neq 0$ and $x^2 > 0$: $1 - x^2 < 1$. Therefore, for any proportion x such that $0 < x < 1$, the capital is multiplied by $1 - x^2$, a number strictly less than 1: the final capital is strictly less than the initial capital. ■

d) No: $(1+x)(1-x) = (1-x)(1+x)$ (multiplication is commutative). In both cases I multiply by $1-x^2$: the order makes no difference.

To remember. For a positive initial capital and a proportion $0 < x < 1$, losing and then regaining this same proportion does not bring you back to the starting point: the multiplier is $1-x^2 < 1$. This is the identity $(a+b)(a-b)$ used in financial reasoning.

Cross-topic block: Justifying and reasoning

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Method. Related models: **D2** (question a)); **D1** (question a)); **F2** (question a)); **T3** (question b)); **T2** (question c)); **L1** (question c)); **J2** (question d)).

a) The first equality is false: in $E = 4x^2 + 9 - (2x^2 + 3x - 2x - 3)$, the square $(2x+3)^2$ is replaced by $4x^2 + 9$. The mistake made is forgetting the cross-term: the square of a sum is not the sum of the squares.

I expand each product separately.

I use $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 2x$ and $b = 3$: $(2x+3)^2 = (2x)^2 + 2 \times 2x \times 3 + 3^2 = 4x^2 + 12x + 9$.

$$\begin{aligned} (x-1)(2x+3) &= x \times 2x + x \times 3 + (-1) \times 2x + (-1) \times 3 \\ &= 2x^2 + 3x - 2x - 3 && \text{I calculate each product} \\ &= 2x^2 + x - 3 && \text{I collect like terms} \end{aligned}$$

$$\begin{aligned} E &= (2x+3)^2 - (x-1)(2x+3) \\ &= (4x^2 + 12x + 9) - (2x^2 + x - 3) && \text{I replace each product with its expansion} \\ &= 4x^2 + 12x + 9 - 2x^2 - x + 3 && \text{the } - \text{ sign changes the sign of each term inside the brackets} \\ &= \boxed{2x^2 + 11x + 12} && \text{I collect like terms} \end{aligned}$$

Check ($x = 1$): $5^2 - 0 \times 5 = 25$ and $2 + 11 + 12 = 25$. ✓

The bracket common to both terms is $(2x+3)$, because $(2x+3)^2 = (2x+3) \times (2x+3)$.

$$\begin{aligned} E &= (2x+3)^2 - (x-1)(2x+3) \\ &= (2x+3)[(2x+3) - (x-1)] && \text{I factor out } (2x+3) \\ &= (2x+3)(2x+3-x+1) && \text{the } - \text{ sign changes the sign of each term inside the brackets} \\ &= \boxed{(2x+3)(x+4)} && \text{I simplify inside the square brackets} \end{aligned}$$

Check: by expanding the factorised form, $(2x+3)(x+4) = 2x^2 + 8x + 3x + 12 = 2x^2 + 11x + 12$, I get the simplified form back. ✓

b) i) For $x = 1$, $(x+5)^2 = 6^2 = 36$ whereas $x^2 + 25 = 1 + 25 = 26$: since $36 \neq 26$, the equality is not true for every number x .

ii) For $x = 1$, $(5x)^2 = 5^2 = 25$ whereas $5x^2 = 5 \times 1 = 5$: since $25 \neq 5$, the equality is not true for every number x .

iii) For $x = 1$, $x^2 - 64 = 1 - 64 = -63$ whereas $(x-8)^2 = (-7)^2 = 49$: since $-63 \neq 49$, the equality is not true for every number x .

iv) For $x = 1$, $x^2 + 16 = 1 + 16 = 17$ whereas $(x+4)^2 = 5^2 = 25$: since $17 \neq 25$, the equality is not true for every number x .

c) i) FALSE. In $3x+2$, the last operation is adding $3x$ and 2 : it is a sum, with terms $3x$ and 2 ; the multiplication $3 \times x$ is inside the term $3x$.

ii) FALSE. Counterexample: $x^2 + 16$. For any number x , $x^2 \geq 0$, so $x^2 + 16 \geq 16$: this sum is never equal to 0 . However, a product $(ax+b)(cx+d)$ of two first-degree expressions, with $a \neq 0$, is equal to 0 for $x = -\frac{b}{a}$, because its first factor is then zero. Therefore $x^2 + 16$ cannot be written as such a product, and the statement is not true for all sums.

iii) TRUE. Read from right to left, the three identities in this chapter turn $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$ and $a^2 - b^2$ into products; none of them transforms the sum $a^2 + b^2$.

d) Let n be any integer: the letter n denotes any integer whatsoever, and I do not replace it with any specific value.

I calculate:

$$\begin{aligned} (n+1)^2 - n^2 &= n^2 + 2n + 1 - n^2 && (a+b)^2 = a^2 + 2ab + b^2 \\ &= 2n + 1 && \text{I collect like terms} \end{aligned}$$

Now n is an integer, so $2n+1$ is twice an integer plus 1 : $2n+1$ is odd.

Therefore, for every integer n , the number $(n+1)^2 - n^2$ is odd. ■

Check: for $n = 4$, $5^2 - 4^2 = 25 - 16 = 9$, odd; for $n = 10$, $11^2 - 10^2 = 121 - 100 = 21$, odd. ✓

Warning! In a), the pupil's second equality contains a second mistake: $-(2x^2+x-3) = -2x^2-x+3$, and not $-2x^2-x-3$. In b), each counterexample disproves "for all", but an equality may still hold for certain values: i), ii) and iv) hold for $x = 0$, and iii) holds for $x = 8$.

6. Brevet-style problems

This bank extends the chapter: problems 1 and 3 to 6 belong to the recommended track (); problems 2 and 7 to 9 are for going further ().

To remember. Each problem in this section comes with its model answer : this write-up contains the elements of a complete solution; other justified write-ups are possible. State the chosen tool (notable identity with its a and b , distributive property or common factor), write down the intermediate line before simplifying, conclude with a full sentence answering the question asked, and check the result (test value, order of magnitude, units digit or re-expanding). Anything written outside the box explains or warns: it should not be copied out.

Method. Related models: **D2** (question a)); **F3** (question b)); **L2** (question c)); **D1** (question c)).

📎 a) I use $(a+b)(a-b) = a^2 - b^2$ with $a = x$ and $b = 6$.

$$(x+6)(x-6) = x^2 - 6^2 \\ = \boxed{x^2 - 36}$$

Check ($x = 1$): $7 \times (-5) = -35$ and $1 - 36 = -35$. ✓

I use $(a-b)(a+b) = a^2 - b^2$ with $a = 2x$ and $b = 5$.

$$(2x-5)(2x+5) = (2x)^2 - 5^2 \\ = \boxed{4x^2 - 25}$$

Check ($x = 1$): $(-3) \times 7 = -21$ and $4 - 25 = -21$. ✓

📎 b) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $x^2 = x^2$ and $256 = 16^2$, so $a = x$ and $b = 16$.

$$x^2 - 256 = x^2 - 16^2 \\ = \boxed{(x+16)(x-16)}$$

Check: $(x+16)(x-16) = x^2 - 16^2 = x^2 - 256$. ✓

Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $64x^2 = (8x)^2$ and $9 = 3^2$, so $a = 8x$ and $b = 3$.

$$64x^2 - 9 = (8x)^2 - 3^2 \\ = \boxed{(8x+3)(8x-3)}$$

Check: $(8x+3)(8x-3) = (8x)^2 - 3^2 = 64x^2 - 9$. ✓

📎 c) The opposite of $4x - 9$ is written $-(4x - 9)$. I change the sign of each of its terms: $-(4x - 9) = \boxed{-4x + 9}$.

Check: $(4x - 9) + (-4x + 9) = 0$. ✓

$$-(6 - x) + 2(x - 1) = -(6 - x) + 2 \times x + 2 \times (-1) \\ = -6 + x + 2x - 2 \\ = \boxed{3x - 8}$$

I calculate each product, and the $-$ sign changes the sign of each term inside the brackets

I collect like terms

Check ($x = 1$): $-5 + 2 \times 0 = -5$ and $3 - 8 = -5$. ✓

Warning! The $-$ sign in front of $(6 - x)$ changes the sign of both terms inside the brackets: $-(6 - x) = -6 + x$. Writing $-6 - x$ means forgetting to change the sign of $-x$.

6.2. Brevet 2: Calculation program

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Method. Related models: **P1** (questions a) and b)); **D2** (question b)); **P2** (question c)).

 a) Starting with 2: $2 + 3 = 5$; $5^2 = 25$; $25 - 2^2 = 25 - 4 = 21$. I do indeed get 21. ✓

 b) Starting with x :

I add 3: $x + 3$.

I square the result: $(x + 3)^2$.

I subtract the square of the starting number: $(x + 3)^2 - x^2$.

The result is $(x + 3)^2 - x^2$. I use $(a + b)^2 = a^2 + 2ab + b^2$ with $a = x$ and $b = 3$:

$$\begin{aligned}(x + 3)^2 - x^2 &= x^2 + 6x + 9 - x^2 & (a + b)^2 &= a^2 + 2ab + b^2 \\ &= \boxed{6x + 9} & \text{I collect like terms}\end{aligned}$$

Check: for $x = 2$, $6 \times 2 + 9 = 21$, the result from question a). ✓

 c) I look for x such that $6x + 9 = 30$.

$$\begin{aligned}6x + 9 &= 30 \\ 6x &= 30 - 9 = 21 & \text{I subtract 9 from both sides} \\ x &= \frac{21}{6} = 3,5 & \text{I divide both sides by 6}\end{aligned}$$

Conclusion: the starting number that gives 30 is $x = 3,5$, and it is the only possible one, since $6x + 9 = 30$ requires $x = 3,5$.

Check: $3,5 + 3 = 6,5$; $6,5^2 = 42,25$; $42,25 - 3,5^2 = 42,25 - 12,25 = 30$. ✓

To remember. Another method in b), using $a^2 - b^2$ with blocks $a = x + 3$ and $b = x$: $[(x + 3) + x][(x + 3) - x] = (2x + 3) \times 3 = 6x + 9$.

6.3. Brevet 3: Expanding and proving a factorisation

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Method. Related models: **D2** (question a)); **D1** (question a)); **J1** (question b)); **V1** (question c)).

 a) I expand each product separately.

I use $(a - b)^2 = a^2 - 2ab + b^2$ with $a = x$ and $b = 3$: $(x - 3)^2 = x^2 - 2 \times x \times 3 + 3^2 = x^2 - 6x + 9$.

$$\begin{aligned}(x - 3)(1 - 2x) &= x \times 1 + x \times (-2x) + (-3) \times 1 + (-3) \times (-2x) \\ &= x - 2x^2 - 3 + 6x & \text{I calculate each product} \\ &= -2x^2 + 7x - 3 & \text{I collect like terms}\end{aligned}$$

$$\begin{aligned}E &= (x - 3)^2 + (x - 3)(1 - 2x) \\ &= (x^2 - 6x + 9) + (-2x^2 + 7x - 3) & \text{I replace each product with its expansion} \\ &= x^2 - 6x + 9 - 2x^2 + 7x - 3 & \text{I remove the brackets preceded by a + sign} \\ &= \boxed{-x^2 + x + 6} & \text{I collect like terms}\end{aligned}$$

Check ($x = 1$): $(-2)^2 + (-2) \times (-1) = 6$ and $-1 + 1 + 6 = 6$. ✓

 b) I expand the proposed form:

$$\begin{aligned}(x - 3)(-x - 2) &= -x^2 - 2x + 3x + 6 & \text{expanding double brackets} \\ &= -x^2 + x + 6 & \text{I collect like terms}\end{aligned}$$

I get the expanded form of E obtained in question a) back. Therefore, for every number x , $E = (x - 3)(-x - 2)$. ■

 c) I choose the factorised form, because for $x = 3$ the factor $x - 3$ is equal to 0: the product is zero without any further calculation. $E = (x - 3)(-x - 2)$.
For $x = 3$:

$$\begin{aligned}E &= (3 - 3) \times (-3 - 2) \\ &= 0 \times (-5) = 0\end{aligned}$$

For $x = 3$, $\boxed{E = 0}$.

To remember. We can also obtain the factorised form directly, since the bracket $(x - 3)$ is common to both terms: $E = (x - 3)[(x - 3) + (1 - 2x)] = (x - 3)(-x - 2)$. To prove the equality, expanding the proposed form is enough.

6.4. Brevet 4: Justified mental calculation

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Method. Related models: **C1** (questions a) and b)); **C2** (question c)).

 a) I use $(a + b)(a - b) = a^2 - b^2$ with $a = 100$ and $b = 4$, because $104 = 100 + 4$ and $96 = 100 - 4$.

$$\begin{aligned}104 \times 96 &= (100 + 4)(100 - 4) = 100^2 - 4^2 \\ &= 10\,000 - 16 = \boxed{9\,984}\end{aligned}$$

I do indeed get $104 \times 96 = 9\,984$. ✓

Check: the units digit is that of $4 \times 6 = 24$, namely 4, and 9 984 is slightly less than $100^2 = 10\,000$. ✓

 b) I use $(a + b)^2 = a^2 + 2ab + b^2$ with $a = 1\,000$ and $b = 2$, because $1\,002 = 1\,000 + 2$.

$$\begin{aligned} 1\,002^2 &= (1\,000 + 2)^2 = 1\,000^2 + 2 \times 1\,000 \times 2 + 2^2 \\ &= 1\,000\,000 + 4\,000 + 4 = \boxed{1\,004\,004} \end{aligned}$$

Check: the units digit is that of $2 \times 2 = 4$, and $1\,004\,004$ is slightly greater than $1\,000^2 = 1\,000\,000$. ✓

 c) I use $a^2 - b^2 = (a + b)(a - b)$ with $a = 57$ and $b = 43$.

$$\begin{aligned} 57^2 - 43^2 &= (57 + 43)(57 - 43) \\ &= 100 \times 14 = \boxed{1\,400} \end{aligned}$$

Check: 57^2 ends like $7 \times 7 = 49$, in 9; 43^2 ends like $3 \times 3 = 9$, in 9; the difference therefore ends in 0, like 1 400. ✓

6.5. Brevet 5: Proof (parity)

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Method. Related models: **J3** (question a)); **J2** (question b)); **T3** (question c)).

 a) $7 + 9 = 16 = 4 \times 4$; $15 + 17 = 32 = 4 \times 8$.

Both sums are multiples of 4.

 b) Let n be any integer. Two consecutive odd numbers can be written as $2n + 1$ and $2n + 3$.

I calculate their sum:

$$\begin{aligned} (2n + 1) + (2n + 3) &= 4n + 4 && \text{I collect like terms} \\ &= 4(n + 1) && \text{I factor out 4} \end{aligned}$$

Now n is an integer, so $n + 1$ is an integer: $4(n + 1)$ is a multiple of 4.

Therefore, for every integer n , the sum of two consecutive odd numbers is a multiple of 4. ■

 c) For the odd numbers 1 and 5, the sum is $1 + 5 = 6$, and 6 is not a multiple of 4 ($4 \times 1 = 4$ and $4 \times 2 = 8$): the statement “the sum of two odd numbers is a multiple of 4” is not true for all odd numbers. No.

Warning! The counterexample in c) must use two odd numbers that are not consecutive: with 1 and 3, the sum 4 is a multiple of 4, as proved in question b).

6.6. Brevet 6: The expression I

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Method. Related models: **D2** (question a)); **D1** (question a)); **F3** (question b)); **V1** (questions c) and d)).

 a) I expand the square separately.

I use $(a - b)^2 = a^2 - 2ab + b^2$ with $a = 7x$ and $b = 3$: $(7x - 3)^2 = (7x)^2 - 2 \times 7x \times 3 + 3^2 = 49x^2 - 42x + 9$.

$$\begin{aligned} I &= (7x - 3)^2 - 25 \\ &= (49x^2 - 42x + 9) - 25 && \text{I replace the square with its expansion} \\ &= \boxed{49x^2 - 42x - 16} && \text{I collect like terms} \end{aligned}$$

Check ($x = 1$): $4^2 - 25 = -9$ and $49 - 42 - 16 = -9$. ✓

 b) Two terms separated by a $-$ sign, and both are squares: this is a difference of two squares. $(7x - 3)^2$ is the square of the block $7x - 3$ and $25 = 5^2$, so $a = 7x - 3$ and $b = 5$.

$$\begin{aligned} (7x - 3)^2 - 25 &= (7x - 3)^2 - 5^2 \\ &= [(7x - 3) + 5] [(7x - 3) - 5] && a^2 - b^2 = (a + b)(a - b) \\ &= \boxed{(7x + 2)(7x - 8)} && \text{I simplify each factor} \end{aligned}$$

Check: $(7x + 2)(7x - 8) = 49x^2 - 56x + 14x - 16 = 49x^2 - 42x - 16$ and $(7x - 3)^2 - 25 = 49x^2 - 42x - 16$ from question a). ✓

 c) Starting form: $I = (7x - 3)^2 - 25$. For $x = 1$:

$$\begin{aligned} I &= (7 \times 1 - 3)^2 - 25 \\ &= 4^2 - 25 = 16 - 25 = -9 \end{aligned}$$

For $x = 1$, $I = -9$.

Expanded form: $I = 49x^2 - 42x - 16$. For $x = 1$:

$$\begin{aligned} I &= 49 \times 1^2 - 42 \times 1 - 16 \\ &= 49 - 42 - 16 = -9 \end{aligned}$$

For $x = 1$, $I = -9$.

Factorised form: $I = (7x + 2)(7x - 8)$. For $x = 1$:

$$\begin{aligned} I &= (7 \times 1 + 2) \times (7 \times 1 - 8) \\ &= 9 \times (-1) = -9 \end{aligned}$$

For $x = 1$, $I = -9$.

All three calculations are correct and give -9 ; I would choose the factorised form, which is the shortest here: a product of two simple numbers.

🔗 d) Starting form: $I = (7x - 3)^2 - 25$. For $x = \frac{3}{7}$:

$$\begin{aligned} I &= \left(7 \times \frac{3}{7} - 3\right)^2 - 25 \\ &= (3 - 3)^2 - 25 = 0^2 - 25 = -25 \end{aligned}$$

For $x = \frac{3}{7}$, $I = -25$.

Factorised form: $I = (7x + 2)(7x - 8)$. For $x = \frac{3}{7}$:

$$\begin{aligned} I &= \left(7 \times \frac{3}{7} + 2\right) \times \left(7 \times \frac{3}{7} - 8\right) \\ &= (3 + 2) \times (3 - 8) = 5 \times (-5) = -25 \end{aligned}$$

For $x = \frac{3}{7}$, $I = -25$.

Both forms give -25 without fractions, because $7 \times \frac{3}{7} = \frac{7 \times 3}{7} = 3$; the starting form is the shortest here: the block $7x - 3$ equals 0.

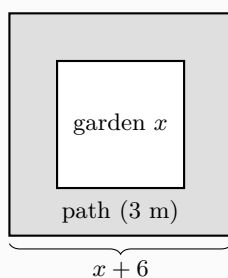
🔗 No: no form is always the best; in c), it was the factorised form. I choose the form according to the value of x and the purpose of the calculation.

6.7. Brevet 7: The garden path 🏡

★★★★

Method. Associated models: **T1** (questions a) and d)); **J1** (question b)); **V1** (question c)).

🔗 a) Diagram first: two nested squares; the 3 m path borders the garden on both sides of each dimension, so the large square has side length $x + 3 + 3 = x + 6$.



🔗 b) I let $\mathcal{A}$ be the area of the path, in m^2 : it is the area of the large square minus the area of the garden, so $\mathcal{A} = (x + 6)^2 - x^2$. I use $a^2 - b^2 = (a + b)(a - b)$ with the blocks $a = x + 6$ and $b = x$.

$$\begin{aligned} (x + 6)^2 - x^2 &= [(x + 6) + x] [(x + 6) - x] & a^2 - b^2 &= (a + b)(a - b) \\ &= (2x + 6) \times 6 & & \text{I simplify each factor} \\ &= 12x + 36 & & \text{distributive property} \\ &= 12(x + 3) & & \text{I factor out 12} \end{aligned}$$

I obtain $12(x + 3)$ as required. Therefore, for all $x > 0$, the area of the path is $12(x + 3) \text{ m}^2$. ■

🔗 c) I choose the factorised form, which can be calculated mentally. $\mathcal{A} = 12(x + 3)$. For $x = 17$:

$$\begin{aligned} \mathcal{A} &= 12 \times (17 + 3) \\ &= 12 \times 20 = 240 \end{aligned}$$

For $x = 17$, $\mathcal{A} = 240 \text{ m}^2$.

Conclusion: the area of the path is $12(x + 3) \text{ m}^2$, that is, 240 m^2 for a garden of side length 17 m.

🔗 d) Check: the path is a strip 3 m wide around a 17 m square; its centre line is about $4 \times 20 = 80 \text{ m}$ all the way round, giving an area of about $3 \times 80 = 240$, that is, 240 m^2 : the answer to question c) is plausible. ✓

6.8. Brevet 8: The product of the two neighbours 🏡

★★★★

Method. Associated models: **J1** (question a)); **C1** (question b)).

🔗 a) I expand the left-hand side. I use $(a - b)(a + b) = a^2 - b^2$ with $a = n$ and $b = 1$.

$$\begin{aligned} (n - 1)(n + 1) + 1 &= n^2 - 1 + 1 & (a - b)(a + b) &= a^2 - b^2 \\ &= n^2 & & \text{I collect like terms} \end{aligned}$$

I obtain the right-hand side. Therefore, for every integer n , $(n - 1)(n + 1) + 1 = n^2$. ■

 b) I apply the equality from a) with $n = 60$, $n = 80$ and then $n = 1\,000$:

$$59 \times 61 = 60^2 - 1 = 3\,600 - 1 = \boxed{3\,599}; \quad 79 \times 81 = 80^2 - 1 = 6\,400 - 1 = \boxed{6\,399}; \quad 999 \times 1\,001 = 1\,000^2 - 1 = 1\,000\,000 - 1 = \boxed{999\,999}.$$

Check: $9 \times 1 = 9$, each product ends in 9; and each is slightly smaller than the square of the middle number. ✓

 c) Conclusion: “the product of two integers immediately either side of the integer n , that is, $n - 1$ and $n + 1$, is equal to $\boxed{n^2 - 1}$, the square of n decreased by 1”.

6.9. Brevet 9: Sum of the squares of two consecutive integers

★★★★

Method. Associated models: **J3** (question a)); **J2** (question b)).

 a) For $n = 0$, $n = 1$, $n = 2$, $n = 3$ and $n = 4$, the sum is: $0^2 + 1^2 = 1$; $1^2 + 2^2 = 5$; $2^2 + 3^2 = 13$; $3^2 + 4^2 = 25$; $4^2 + 5^2 = 41$.

These five sums are odd: it seems that the sum of the squares of two consecutive integers is always odd.

These examples only establish the tested cases: the statement is about infinitely many integers, so a proof using a letter is required.

 b) Let n be any integer. Two consecutive integers can be written as n and $n + 1$.

I calculate the sum of their squares. I use $(a + b)^2 = a^2 + 2ab + b^2$ with $a = n$ and $b = 1$ to expand $(n + 1)^2$:

$$\begin{aligned} n^2 + (n + 1)^2 &= n^2 + n^2 + 2n + 1 & (a + b)^2 &= a^2 + 2ab + b^2 \\ &= 2n^2 + 2n + 1 & & \text{I collect like terms} \\ &= 2(n^2 + n) + 1 & & \text{I factor out 2 from } 2n^2 + 2n \end{aligned}$$

Now n is an integer, so $k = n^2 + n$ is an integer: $2(n^2 + n) + 1 = 2k + 1$ is an odd number.

Check: for $n = 4$, $16 + 25 = 41 = 2 \times 20 + 1$, with $k = 4^2 + 4 = 20$. ✓

Therefore, for every integer n , $n^2 + (n + 1)^2$ is $\boxed{\text{odd}}$. ■

 c) Writing it in the form $2(n^2 + n) + 1$ is what shows that it is odd: the “ $2 \times$ integer” part makes it even, and the “ $+1$ ” breaks the evenness. I can also say that $2n^2$ and $2n$ are even: their sum, increased by 1, is odd. Factorising brings this justification together into a single form $2k + 1$.

7. Cross-curricular exercises in context

7.1. Physics: Braking energy

★★★☆

Method. Associated model: **V1** (questions a) and b)).

 a) $E_c = \frac{1}{2} m v^2$. For $m = 1\,200$ and $v = 10$:

$$\begin{aligned} E_c &= \frac{1}{2} \times 1\,200 \times 10^2 \\ &= 600 \times 100 = 60\,000 \end{aligned}$$

For a 1 200 kg car travelling at 10 m/s, $\boxed{E_c = 60\,000 \text{ J}}$.

Check: half of 1 200 is 600, and $10^2 = 100$; $600 \times 100 = 60\,000$. ✓

 b) $E_c = \frac{1}{2} m v^2$. For $m = 1\,200$ and $v = 20$:

$$\begin{aligned} E_c &= \frac{1}{2} \times 1\,200 \times 20^2 \\ &= 600 \times 400 = 240\,000 \end{aligned}$$

At 20 m/s, $\boxed{E_c = 240\,000 \text{ J}}$.

Karim’s mistake: $(2v)^2 = 4v^2$, and not $2v^2$, because the square applies to the whole factor.

Conclusion: when the speed doubles, the kinetic energy is multiplied by $240\,000 \div 60\,000 = \boxed{4}$, not by 2.

 c) If v becomes kv , the new kinetic energy is:

$$\begin{aligned} \frac{1}{2} m (kv)^2 &= \frac{1}{2} m k^2 v^2 & (kv)^2 &= k^2 v^2: \text{the square applies to the whole factor} \\ &= k^2 \times \frac{1}{2} m v^2 & & \text{multiplication is commutative} \end{aligned}$$

The energy is multiplied by $\boxed{k^2}$. This matches the “Common mistakes” box in the course notes: writing $(2x)^2 = 2x^2$ is an error, because the square applies to the whole factor; here, $(kv)^2 = k^2 v^2$.

Check: for $k = 2$, I recover the factor of 4 from question b). ✓

7.2. Land planning: The reshaped plot

★★★☆☆

Method. Associated models: **T3** (question a)); **D2** (question b)).

 a) For $a = 20$: the square has area $20^2 = 400$, that is, $\boxed{400 \text{ m}^2}$, and the rectangle $25 \times 15 = 375$, that is, $\boxed{375 \text{ m}^2}$. The areas are not equal: the mayor is already wrong on this example, and the difference is $400 - 375 = 25 \text{ m}^2$.

 b) I use $(A + B)(A - B) = A^2 - B^2$ with $A = a$, the side length of the square, and $B = 5$.

$$\begin{aligned}(a + 5)(a - 5) &= a^2 - 5^2 \\ &= a^2 - 25\end{aligned}$$

Check ($a = 20$): $400 - 25 = 375$, the area found in a). ✓

The area of the rectangle is $a^2 - 25$. Whatever the value of a , the rectangle has exactly $\boxed{25 \text{ m}^2}$ less than the square a^2 .

 c) Conclusion: “the loss is always 25 m^2 , whether the plot has side length 20 m or 2 km; it does not depend on the size of the plot, only on the 5 m shift”.

7.3. Programming: The identity tester

★★★☆☆

Method. Associated models: **J3** (questions b) and c)); **J1** (question c)).

 a) For each x from -5 to 5 , the script displays x , the value of $(x + 5)^2 - x^2$ and that of $10x + 25$: $x = -5$: -5 -25 -25 ; $x = 0$: 0 25 25 ; $x = 2$: 2 45 45 .

 b) It seems that, for every number x , $(x + 5)^2 - x^2 = 10x + 25$.

 c) $\boxed{\text{No}}$: eleven tested values prove nothing for the others.

I expand the left-hand side. I use $(a + b)^2 = a^2 + 2ab + b^2$ with $a = x$ and $b = 5$.

$$\begin{aligned}(x + 5)^2 - x^2 &= x^2 + 10x + 25 - x^2 & (a + b)^2 &= a^2 + 2ab + b^2 \\ &= \boxed{10x + 25} & \text{I collect like terms}\end{aligned}$$

I obtain the right-hand side. Therefore, for every number x , $(x + 5)^2 - x^2 = 10x + 25$. ■

 d) In the loop, I replace the display line with: `print(x, (x+5)**2, x**2+25)`. Only $\boxed{x = 0}$ displays two equal numbers (25 and 25), because the double product $10x$ is zero only at 0.

To remember. Another route in c), using $a^2 - b^2$: $(x + 5)^2 - x^2 = (2x + 5) \times 5 = 10x + 25$.

7.4. History of calculation: Horner’s machine

★★★☆☆

Method. Associated models: **V1** (questions a) and d)); **J1** (question b)).

 a) Machine, for $x = 2$:

$$2 \xrightarrow{\times 2} 4 \xrightarrow{+5} 9 \xrightarrow{\times 2} 18 \xrightarrow{-7} 11 \xrightarrow{\times 2} 22 \xrightarrow{+8} \boxed{30}.$$

Naive route: $B = 2x^3 + 5x^2 - 7x + 8$. For $x = 2$:

$$\begin{aligned}B &= 2 \times 2^3 + 5 \times 2^2 - 7 \times 2 + 8 \\ &= 16 + 20 - 14 + 8 = 30\end{aligned}$$

For $x = 2$, $\boxed{B = 30}$. Both routes agree.

 b) I expand the left-hand side, for every number x .

$$\begin{aligned}((2x + 5)x - 7)x + 8 &= (2x^2 + 5x - 7)x + 8 & \text{distributive property} \\ &= 2x^3 + 5x^2 - 7x + 8 & \text{distributive property}\end{aligned}$$

I obtain the right-hand side. Therefore, for every number x , $((2x + 5)x - 7)x + 8 = 2x^3 + 5x^2 - 7x + 8$. ■ The machine therefore calculates B exactly, whatever the value of x : it is an identity.

 c) Naive route: $2 \times x \times x \times x$ (three multiplications), $5 \times x \times x$ (two), $7 \times x$ (one): $\boxed{6}$ multiplications in all. Machine: $\times 2$, $\times x$, $\times x$: $\boxed{3}$ multiplications, half as many, without losing anything (the identity in b) guarantees this).

 d) $C = 3x^2 - 4x + 3 = (3x - 4)x + 3$ (I factor out x from $3x^2 - 4x$). $C = (3x - 4)x + 3$. For $x = 10$:

$$\begin{aligned}C &= (3 \times 10 - 4) \times 10 + 3 \\ &= (30 - 4) \times 10 + 3 = 260 + 3 = 263\end{aligned}$$

For $x = 10$, $\boxed{C = 263}$, mentally: the nested form is also a clever calculation form.

8. Going further

8.1. Challenge: An almost prime square

★★★★

Method. Associated model: **F3**.

 I factorise. Two terms separated by a $-$ sign, and both are squares: it is a difference of two squares. $n^2 = n^2$ and $9 = 3^2$, so $a = n$ and $b = 3$.

$$\begin{aligned} n^2 - 9 &= n^2 - 3^2 \\ &= (n - 3)(n + 3) \quad a^2 - b^2 = (a - b)(a + b) \end{aligned}$$

Check: $(n - 3)(n + 3) = n^2 - 3^2 = n^2 - 9$. ✓

For $n \leq 3$, $n^2 - 9 \leq 0$ is not prime. For $n \geq 4$, both factors are strictly positive integers; if their product is prime, one of them must equal 1 (otherwise it has a divisor other than 1 and itself). Since $n + 3 \geq 3$, the only possibility is $n - 3 = 1$, that is, $n = 4$. Conversely, for $n = 4$, $n^2 - 9 = 16 - 9 = 7$, which is prime.

Conclusion: $\boxed{n = 4}$ is the unique solution.

8.2. Proof without words: The sum of odd numbers

★★★★

Method. Associated model: **J1** (question c)).

 a) $1 + 3 + 5 + 7 + 9 = \boxed{25} = 5^2$; $1 + 3 + \dots + 19 = \boxed{100} = 10^2$.

 b) For every integer $n \geq 1$, the sum of the first n positive odd integers is equal to n^2 . Each L-shaped border adds the next layer: an $n \times n$ square is made up of exactly n nested L-shapes, with areas $1, 3, 5, \dots, 2n - 1$.

 c) The area of the n -th L-shape is what I add to go from the square $(n - 1)^2$ to the square n^2 . I use $a^2 - b^2 = (a + b)(a - b)$ with $a = n$ and $b = n - 1$.

$$\begin{aligned} n^2 - (n - 1)^2 &= [n + (n - 1)][n - (n - 1)] & a^2 - b^2 &= (a + b)(a - b) \\ &= (2n - 1) \times 1 & \text{I simplify each factor} \\ &= 2n - 1 \end{aligned}$$

This is indeed the n -th odd number.

Therefore, by stacking the L-shapes from 1 to n , I fill the square: for every integer $n \geq 1$, $\boxed{1 + 3 + \dots + (2n - 1) = n^2}$. ■

8.3. Prep. for Year 11: The conjugate expression

★★★★

Method. Associated model: **D2** (question a)).

 a) I use $(a - b)(a + b) = a^2 - b^2$ with $a = 4$ and $b = \sqrt{7}$.

$$\begin{aligned} (4 - \sqrt{7})(4 + \sqrt{7}) &= 4^2 - (\sqrt{7})^2 \\ &= 16 - 7 = \boxed{9} \quad (\sqrt{7})^2 = 7 \end{aligned}$$

 b) The product calculated in a) equals $9 \neq 0$, so both of its factors $4 - \sqrt{7}$ and $4 + \sqrt{7}$ are non-zero. The fraction is well-defined, and I can multiply its numerator and denominator by $4 + \sqrt{7}$:

$$\begin{aligned} B &= \frac{1}{4 - \sqrt{7}} \\ &= \frac{1 \times (4 + \sqrt{7})}{(4 - \sqrt{7})(4 + \sqrt{7})} & \text{I multiply the numerator and denominator by } 4 + \sqrt{7} \\ &= \frac{\boxed{4 + \sqrt{7}}}{9} & \text{question a)} \end{aligned}$$

Check: $4 - \sqrt{7} \approx 1,35$, so $B \approx 0,74$; and $\frac{4 + 2,65}{9} \approx 0,74$. ✓

 c) It is $a^2 - b^2 = (a - b)(a + b)$ that “makes everything work”: it turns the denominator into $4^2 - (\sqrt{7})^2 = 16 - 7$, an integer, because $(\sqrt{7})^2 = 7$. Same idea as $102 \times 98 = (100 + 2)(100 - 2)$: I pair an awkward number into a “sum $\times$ difference” to make the troublesome part disappear.

8.4. The hidden perfect square

★★★★

Method. Associated models: **D1** (question a)); **J1** (question b)).

 a)

$$\begin{aligned} a &= n(n+3) = n \times n + n \times 3 \\ &= n^2 + 3n && \text{I calculate each product} \\ (n+1)(n+2) &= n \times n + n \times 2 + 1 \times n + 1 \times 2 \\ &= n^2 + 2n + n + 2 && \text{I calculate each product} \\ &= n^2 + 3n + 2 && \text{I collect like terms} \end{aligned}$$

Check ($n=1$): $1 \times 4 = 4$ and $1 + 3 = 4$; $2 \times 3 = 6$ and $1 + 3 + 2 = 6$. ✓

Therefore $(n+1)(n+2) = (n^2 + 3n) + 2 = \boxed{a+2}$.

 b) I group the first factor with the last, and the two middle factors together.

$$\begin{aligned} n(n+1)(n+2)(n+3) + 1 &= \underbrace{[n(n+3)]}_{=a} \times \underbrace{[(n+1)(n+2)]}_{=a+2} + 1 && \text{multiplication is commutative} \\ &= a(a+2) + 1 && \text{question a)} \\ &= a^2 + 2a + 1 && \text{distributive property} \\ &= \boxed{(a+1)^2} && A^2 + 2AB + B^2 = (A+B)^2 \text{ with } A=a \text{ and } B=1 \end{aligned}$$

I obtain $(a+1)^2$ as required. Therefore, for every natural number n , $n(n+1)(n+2)(n+3) + 1 = a(a+2) + 1 = (a+1)^2$. ■

 c) The product of four consecutive integers, increased by 1, is always a perfect square: it is the square of $a+1 = \boxed{n^2 + 3n + 1}$.

Check for $n=3$: $3 \times 4 \times 5 \times 6 + 1 = 360 + 1 = 361$, and $n^2 + 3n + 1 = 9 + 9 + 1 = 19$; and indeed $19^2 = 361$. ✓

8.5. Giant prime factors (echo of 8.1)

★★★★

Method. Associated models: **C2** (questions a) and b)); **F3** (question b)); **J1** (question c)).

 a) I use $a^2 - b^2 = (a-b)(a+b)$ with $a = 5\,000$ and $b = 1$.

$$\begin{aligned} A &= 24\,999\,999 = 25\,000\,000 - 1 = 5\,000^2 - 1^2 \\ &= (5\,000 - 1)(5\,000 + 1) && a^2 - b^2 = (a-b)(a+b) \\ &= 4\,999 \times 5\,001 \end{aligned}$$

Check: $4\,999 \times 5\,001$ ends like $9 \times 1 = 9$, in 9, just like $24\,999\,999$. ✓

Now $5\,001 = 3 \times 1\,667$ (sum of digits $5 + 0 + 0 + 1 = 6$, a multiple of 3). Hence

$$\boxed{A = 3 \times 1\,667 \times 4\,999}$$

(1 667 and 4 999 are prime, assumed).

 b) $10^6 + 18 \times 10^3 + 81 = 1\,000\,000 + 18\,000 + 81 = 1\,018\,081 = B$. ✓

Three terms, and the two outer ones are squares: it may be a perfect square. $10^6 = (10^3)^2$ and $81 = 9^2$, so $a = 10^3$ and $b = 9$; the double product is $2 \times 10^3 \times 9 = 18\,000$: this is indeed the middle term, preceded by a + sign.

$$\begin{aligned} B &= (10^3)^2 + 2 \times 10^3 \times 9 + 9^2 \\ &= (10^3 + 9)^2 = 1\,009^2 && a^2 + 2ab + b^2 = (a+b)^2 \end{aligned}$$

Check: $(10^3 + 9)^2 = (10^3)^2 + 2 \times 10^3 \times 9 + 9^2 = 10^6 + 18 \times 10^3 + 81$. ✓

Therefore $\boxed{B = 1\,009^2}$ (1 009 is prime, assumed): the prime factorisation of B is $1\,009 \times 1\,009$.

 c) I expand the right-hand side.

$$\begin{aligned} (n-2)(n^2 + 2n + 4) &= n^3 + 2n^2 + 4n - 2n^2 - 4n - 8 && \text{expanding double brackets} \\ &= n^3 - 8 && \text{I collect like terms} \end{aligned}$$

I obtain the left-hand side. Therefore, for every integer n , $n^3 - 8 = (n-2)(n^2 + 2n + 4)$. ■

For $n \leq 2$, $n^3 - 8 \leq 0$ is not prime. For $n \geq 3$, both factors are strictly positive integers: if their product is prime, one of them must equal 1 (same argument as in 8.1). Now $n^2 + 2n + 4 = (n+1)^2 + 3 \geq 3$: this factor never equals 1. This leaves $n-2 = 1$, that is, $n = 3$. Conversely, for $n = 3$, $n^3 - 8 = 27 - 8 = 19$, which is prime.

Conclusion: $\boxed{n = 3}$ is the unique solution, and $n^3 - 8 = 19$.

8.6. The Pythagorean triple machine

★★★★

Method. Associated model: **J1** (question c)).

 **a)** $(u, v) = (1, 2)$: $a = 2^2 - 1^2 = 3$, $b = 2 \times 1 \times 2 = 4$, $c = 2^2 + 1^2 = 5 \rightarrow (3, 4, 5)$.
 $(2, 3)$: $a = 9 - 4 = 5$, $b = 12$, $c = 13 \rightarrow (5, 12, 13)$. $(3, 4)$: $a = 16 - 9 = 7$, $b = 24$, $c = 25 \rightarrow (7, 24, 25)$.
 $(2, 5)$: $a = 25 - 4 = 21$, $b = 20$, $c = 29 \rightarrow (21, 20, 29)$.

 **b)** $3^2 + 4^2 = 9 + 16 = 25 = 5^2$. ✓

 **c)** I expand each expression.

$$\begin{aligned} (v^2 - u^2)^2 + (2uv)^2 &= (v^4 - 2u^2v^2 + u^4) + 4u^2v^2 & (A - B)^2 &= A^2 - 2AB + B^2 \text{ and } (2uv)^2 = 4u^2v^2 \\ &= v^4 + 2u^2v^2 + u^4 & \text{I collect like terms} \\ (v^2 + u^2)^2 &= v^4 + 2u^2v^2 + u^4 & (A + B)^2 &= A^2 + 2AB + B^2 \end{aligned}$$

Both expressions are equal to $v^4 + 2u^2v^2 + u^4$. Since u and v are integers with $0 < u < v$, a , b and c are strictly positive integers. Therefore, for all integers u and v such that $0 < u < v$, $\boxed{a^2 + b^2 = c^2}$: (a, b, c) is a Pythagorean triple. ■

 **d)** For example $(u, v) = (4, 7)$: $a = 49 - 16 = 33$, $b = 56$, $c = 49 + 16 = 65$; and $33^2 + 56^2 = 1\,089 + 3\,136 = 4\,225 = 65^2$. ✓

To remember. The three identities of the chapter are not just for calculating quickly: here, a single expansion produces infinitely many right-angled triangles with integer sides.

9. The bridge to chapter 02

9.1. Factorise... then solve?

★★☆☆

Method. Associated models: **V1** (questions a) and c)); **F3** (question b)).

 **a)** The left-hand side of the equation is $x^2 - 9$. For $x = 3$:

$$\begin{aligned} x^2 - 9 &= 3^2 - 9 \\ &= 9 - 9 = 0 \end{aligned}$$

For $x = 3$, $x^2 - 9 = 0$: $\boxed{\text{the number 3 works}}$.

It is $\boxed{\text{not necessarily}}$ the only one: two opposite numbers have the same square, and nothing yet proves that no other number works.

 **b)** Two terms separated by a $-$ sign, and both are squares: it is a difference of two squares. $x^2 = x^2$ and $9 = 3^2$, so $a = x$ and $b = 3$.

$$\begin{aligned} x^2 - 9 &= x^2 - 3^2 \\ &= \boxed{(x + 3)(x - 3)} & a^2 - b^2 &= (a + b)(a - b) \end{aligned}$$

Check: $(x + 3)(x - 3) = x^2 - 3^2 = x^2 - 9$. ✓

 **c)** I look for a number that makes the factor $x + 3$ equal to zero: $-3 + 3 = 0$, and the product $(x + 3)(x - 3)$ then equals $0 \times (-6) = 0$. I therefore choose -3 . The left-hand side of the equation is $x^2 - 9$. For $x = -3$:

$$\begin{aligned} x^2 - 9 &= (-3)^2 - 9 \\ &= 9 - 9 = 0 \end{aligned}$$

For $x = -3$, $x^2 - 9 = 0$: $\boxed{\text{the number } -3 \text{ works}}$.

 **d)** What has been checked: 3 and -3 are solutions to $x^2 - 9 = 0$, by the calculations in questions a) and c).

What is conjectured: it seems that, if a product of two numbers equals 0, at least one of those two numbers is zero. With $(x + 3)(x - 3) = 0$, I would then obtain $x + 3 = 0$ or $x - 3 = 0$, so $x = -3$ or $x = 3$: there would be no third solution.

What remains to be proved: this zero-product property. It has not yet been established in this chapter: factorising is therefore $\boxed{\text{not yet}}$ enough to prove that there is no third solution. The property will be proved in chapter 02.

Method. Associated models: **F3** (question a)); **V1** (question b)).

 **a)** Ninon's method: $x - 1 = 3$ gives $x = 4$; $x - 1 = -3$ gives $x = -2$.

Grégory's method: $(x - 1)^2 - 9 = 0$; I factorise the left-hand side. Two terms separated by a $-$ sign, and both are squares: it is a difference of two squares. $(x - 1)^2$ is the square of the block $x - 1$ and $9 = 3^2$, so $a = x - 1$ and $b = 3$.

$$\begin{aligned}(x - 1)^2 - 9 &= (x - 1)^2 - 3^2 \\ &= [(x - 1) - 3][(x - 1) + 3] \quad a^2 - b^2 = (a - b)(a + b) \\ &= (x - 4)(x + 2) \quad \text{I simplify each factor}\end{aligned}$$

Check: $(x - 4)(x + 2) = x^2 + 2x - 4x - 8 = x^2 - 2x - 8$ and $(x - 1)^2 - 9 = x^2 - 2x + 1 - 9 = x^2 - 2x - 8$. ✓

Hence $(x - 4)(x + 2) = 0$: each of the values 4 and -2 makes one factor equal to zero. Both forms bring out the same values 4 and -2 .

 **b)** The left-hand side of the equation is $(x - 1)^2$. For $x = 4$:

$$\begin{aligned}(x - 1)^2 &= (4 - 1)^2 \\ &= 3^2 = 9\end{aligned}$$

For $x = 4$, $(x - 1)^2 = 9$: the number 4 works.

The left-hand side of the equation is $(x - 1)^2$. For $x = -2$:

$$\begin{aligned}(x - 1)^2 &= (-2 - 1)^2 \\ &= (-3)^2 = 9\end{aligned}$$

For $x = -2$, $(x - 1)^2 = 9$: the number -2 works.

Factorising is not yet enough to prove that there is no other solution: the zero-product property is missing. Ninon's method does not prove it either: she assumes that 3 and -3 are the only numbers whose square is 9, which is precisely the question left open in 9.1.

 **c)** Grégory reuses the factorisation from the chapter ("nested" difference of two squares, as in 3.4). To use $(x - 4)(x + 2) = 0$ rigorously and conclude that one of the factors is zero, we still need the zero-product property: it will be stated and proved in chapter 02.

End of the worked solutions. The  boxes provide justified model solutions. The  exercises remain optional.