

Chapter 1B: Algebraic expressions

Expand, factorise, prove: the notable identities



Starting point

In Year 9, you learnt how to expand $k(a + b)$ and how to simplify an expression. This chapter crosses three thresholds:

- expanding double brackets (double distributive property) $(a + b)(c + d)$, which we will prove rather than take for granted;
- the three notable identities, proved both by calculation and by geometry: three formulas that make mental calculation fast (201^2 or $84^2 - 16^2$ in your head) and make it possible to factorise;
- using letters to prove a result that is true for all numbers: “the sum of five consecutive integers is always a multiple of 5” can be checked on examples, but its general proof requires reasoning that holds for every integer.

Where this leads: in Chapter 02, after the direct intercept theorem, factorising will unlock solving equations such as $x^2 - 16 = 0$ (zero-product equations); in the brevet (French national exam at the end of Year 10), calculation programs and factorising $a^2 - b^2$ are classics; in upper secondary school, these identities are used in particular to transform expressions with square roots, to complete the square and, later on, to differentiate functions.

No computational virtuosity is expected: the difficulty of this chapter is never the length of the calculations. You need to recognise the main operation, justify a transformation and check that an equality remains true. The formulas will be built step by step; do not memorise them before you have understood them.

Fluency skills

To be done mentally, without written working, then repeated from memory; the upside-down answers below are for self-marking. Questions 4 to 6 and 10 review previous chapters.

1. Expand $3(x - 4)$.

2. Simplify $6x + 5 - 4x + 2$.

3. Give the opposite of $7 - 2x$.

4. Calculate $2^3 \times 2^4$.

5. Write $\frac{42}{56}$ in lowest terms.

6. Write 0,0035 in standard form (scientific notation).

7. Calculate $(-5)^2$ then -5^2 .

8. Factorise $4x + 28$.

9. Calculate $x^2 - 3x$ for $x = -2$.

10. Two similar triangles: what can be said about their angles?

Answers, for self-marking: 1) $3x - 12$ 2) $2x + 7$ 3) $-7 + 2x$ 4) $2^7 = 128$ 5) $\frac{7}{8}$ 6) 3.5×10^{-3} 7) 25 and -25 8) $4(x + 7)$ 9) (6) 10) $(-2) \times 3 = -6$ and $9 + 4 = 13$

→ Exercise sheet: “Quick review” block (previous chapters)

Introductory activity: the growing square

Situation. A market gardener has a square vegetable patch with side length 10 m. He extends it by 2 m in both directions: the patch remains square, with side length 12 m.

1. First, without any numbers. Watch the side of the patch grow from a to $a + b$ (here, from 10 to 12): how do you see the figure grow? On a sketch, show where the gained area is located, colour it in, then split it into squares and rectangles. For your way of seeing it, write an expression for the area of the large square. (There are several ways of splitting it: each gives a different expression; keep yours.)

2. Karim claims: “The area gains $2^2 = 4 \text{ m}^2$, since each side gains 2 m.” Calculate the old area and the new area. Is Karim right?

3. Copy and complete the following table:

a	b	$(a + b)^2$	$a^2 + b^2$	gap between the two
10	2			
3	4			
5	1			
7	10			

4. What do you notice? Does the equality $(a + b)^2 = a^2 + b^2$ seem to be true? Compare each gap with the product $a \times b$: what conjecture can you make about the exact value of $(a + b)^2$?

5. Take a square of side length $a + b$ again, without choosing values for a and b . Split it or label it so as to show its entire area; justify the dimensions of each piece. Then compare this splitting with another way of organising the surface: what relationship is there between the two area expressions obtained, and how does each one explain the term observed in the table?

Summary, to be completed after comparison and proof.

The conjecture:

What proves or refutes it for all values:

2

1 The status of the letter: sum or product?

Algebraic expression, variable, constant and numerical value

An algebraic expression is a sequence of operations involving numbers and letters that represent numbers. Within the same expression, a given letter always stands for the same number.

When a letter can take several values, it is a variable. When it stands for a fixed value in the situation, it is a constant. Evaluating an expression, or calculating its numerical value, consists in replacing each letter with the chosen value and then carrying out the calculations while respecting the order of operations.

In plain words: an algebraic expression is a calculation on hold; as soon as we give a value to the letter, it becomes a number.

Why write a negative value in brackets?

Proof tool: direct proof, from the definition of the numerical value.

A letter stands for a number: replacing it with its value changes neither the operations in the expression nor their order. The implied $\times$ sign therefore reappears: $3x$ means $3 \times x$, and for $x = -4$ we write $3 \times (-4)$. It is the entire number, sign included, that replaces the letter: for $x = -4$, x^2 becomes $(-4)^2 = (-4) \times (-4) = 16$. Without brackets, we would write -4^2 , which reads “the opposite of the square of 4” and equals -16 : the square would apply only to 4, and it would no longer be the value of x^2 for $x = -4$. ■

Writing model V1: calculating the value of an expression for a given value

The steps

① Copy the expression without transforming it.

② Replace each occurrence of the letter with its value; a negative value is written in brackets, and the implied $\times$ sign reappears.

③ Carry out the powers, then the multiplications, then the additions and subtractions.

④ Conclude with a sentence that answers the question.

An expression given in several forms is evaluated using the fastest form: see the end of part 4.

On your paper

Calculate $A = (2x + 5)^2 - 3x$ for $x = -1$.

① $A = (2x + 5)^2 - 3x$. ② For $x = -1$:

$$A = (2 \times (-1) + 5)^2 - 3 \times (-1) \quad \text{②}$$
$$= 3^2 + 3 = 9 + 3 = 12 \quad \text{③}$$

④ For $x = -1$, $A = 12$.

→ Exercise sheet: exercises 6.3 c), 6.6 c) and d) and 9.1 a) and c), at the end of the chapter (model V1)

Quick questions (unaided)

1. Calculate $A = 2x^2 - 5x + 1$ for $x = -3$.

2. Calculate $B = u(v + 2u) - v$ for $u = 2$ and $v = 5$.

3. In the formula $P = 2\pi R$, which letters can vary when we change the circle? Which symbol stands for a constant?

Answers: 1) $2 \times (-3)^2 - 5 \times (-3) + 1 = 2 \times 9 + 15 + 1 = 34$; 2) $2 \times (5 + 2 \times 2) - 5 = 2 \times 9 - 5 = 13$; 3) P and R vary, whereas π stands for a constant.

Sum and product: the nature of an expression

After respecting brackets and the order of operations, the main operation is the one that applies to the entire expression. It determines the nature of the expression:

- if the main operation is an addition or a subtraction, the expression is a sum;
- if the main operation is a multiplication, the expression is a product.

Where needed, subtractions are rewritten as additions of opposites: in $4 - 9x = 4 + (-9x)$, the terms are 4 and $-9x$. The expressions thus added at the main level are the terms; those multiplied at the main level are the factors. A square reads as a product of two identical factors: $E^2 = E \times E$. Similarly, a quotient by a number reads as a product: $\frac{x}{2} = \frac{1}{2} \times x$.

It reads: E^2 is “ E squared”; $3x$ is “three x ”: it is the product $3 \times x$, the $\times$ sign being implied between a number and a letter, between two letters or in front of a bracket. In plain words: to find the nature of an expression, look for the last operation you would carry out when evaluating it; that operation gives the expression its name.

Why this classification decides everything

Expanding means going from a product to a sum, and factorising means going from a sum to a product. Before any calculation, ask yourself: “what do I have, and what do I want?”

$$\text{PRODUCT} \xrightarrow{\text{expand}} \text{SUM} \qquad \text{SUM} \xrightarrow{\text{factorise}} \text{PRODUCT}$$

The arrow reads “gives, by expanding” then “gives, by factorising”: in both directions, the value of the expression does not change, only its form changes. Thus $6(x+1)$ is a product and $6x+6$ is a sum; the distributive property nevertheless ensures that these two forms have the same value for every x .

Writing model L1: stating whether an expression is a sum or a product

The steps

- ① Identify the main operation: the one that applies to the entire expression, that is, the last calculation you would carry out; a square E^2 reads as $E \times E$.
 - ② Conclude: addition or subtraction, the expression is a sum; multiplication, it is a product.
 - ③ Name its terms, each with its sign, or its factors.
- A $-$ sign or a multiplication inside brackets is not the main operation.

On your paper

State whether each expression is a sum or a product, then name its terms or its factors: a) $8x+3$; b) $9-4x$; c) $4(x-6)$; d) $(3x+4)^2$; e) $7x^2$; f) $2(x+3)+x(x-1)$.

a) ① In $8x+3$, the last calculation is the addition of $8x$ and 3: ② it is a sum, ③ with terms $8x$ and 3.

b) ① In $9-4x$, the last calculation is the subtraction of $4x$: ② it is a sum, ③ with terms 9 and $-4x$.

c) ① In $4(x-6)$, the last calculation is the multiplication of 4 by $(x-6)$: ② it is a product, ③ with factors 4 and $(x-6)$.

d) ① In $(3x+4)^2$, which reads as $(3x+4) \times (3x+4)$, the last calculation is the multiplication of $(3x+4)$ by $(3x+4)$: ② it is a product, ③ with factors $(3x+4)$ and $(3x+4)$.

e) ① In $7x^2$, which reads as $7 \times x \times x$, the last calculation is a multiplication: ② it is a product, ③ with factors 7, x and x .

f) ① In $2(x+3)+x(x-1)$, the last calculation is the addition of $2(x+3)$ and $x(x-1)$: ② it is a sum, ③ with terms $2(x+3)$ and $x(x-1)$.

→ Exercise sheet: exercise 1.1 (model L1)

Opposite of an expression

Two expressions are opposites when their sum equals 0 for all values of the letter. The opposite of an expression A is written $-A$; when the expression has several terms, it is placed in brackets: the opposite of $4x-11$ is written $-(4x-11)$.

It reads: $-(4x-11)$ is “the opposite of $4x-11$ ”. In plain words: the opposite of an expression is what must be added to it to obtain 0; for a number, it is the same number with the opposite sign, such as 7 and -7 .

Opposite of an expression

The opposite of an expression is obtained by changing the sign of each of its terms:

$$-(a+b) = -a-b \quad \text{and} \quad -(a-b) = -a+b.$$

Examples: $-(2-9x) = -2+9x$; $-(6x-1) = -6x+1$.

It reads: $-(a+b)$ is “the opposite of the sum of a and b ”; the $-$ sign placed in front of the brackets is a factor -1 that multiplies each term. In plain words: taking the opposite means changing every sign, not just the first one.

Why do we change the sign of each term?

Proof tool: direct proof, from the definition of the opposite.

The sum of $a+b$ and $-a-b$ equals $(a+b)+(-a-b) = a-a+b-b = 0$: by definition, $-a-b$ is therefore the opposite of $a+b$. Similarly, $(a-b)+(-a+b) = a-a-b+b = 0$, so $-a+b$ is the opposite of $a-b$. A forgotten sign breaks this zero sum: $(4x-11)+(-4x-11) = -22$, and not 0; that is why the check in model L2 adds the expression and its opposite. ■

Writing model L2: giving the opposite of an expression

The steps

- ① Write the opposite with a $-$ sign in front of the expression placed in brackets.
- ② Change the sign of each of its terms: $-(a + b) = -a - b$ and $-(a - b) = -a + b$.
- ③ Check: the sum of the expression and its opposite equals 0.

On your paper

Give the opposite of each expression: a) $7x - 2$; b) $-x^2 + 4x - 3$.

a) ① The opposite of $7x - 2$ is written $-(7x - 2)$. ② I change the sign of each of its terms: $-(7x - 2) = \boxed{-7x + 2}$.

③ Check: $(7x - 2) + (-7x + 2) = 0$. ✓

b) ① The opposite of $-x^2 + 4x - 3$ is written $-(-x^2 + 4x - 3)$. ② I change the sign of each of its terms: $-(-x^2 + 4x - 3) = \boxed{x^2 - 4x + 3}$.

③ Check: $(-x^2 + 4x - 3) + (x^2 - 4x + 3) = 0$. ✓

→ Exercise sheet: exercises 1.2 b) and 6.1 c) (model L2)

Writing “even”, “odd”, “multiple of”, “consecutive”

Let n be an integer. Then:

- $2n$ is an even integer, and every even integer can be written in this way;
- $2n + 1$ is an odd integer, and every odd integer can be written in this way;
- a multiple of 3 is written $3k$ (with k an integer); a multiple of 7 is written $7k$; etc.
- consecutive integers are written $n, n + 1, n + 2, \dots$; two consecutive odd integers are written $2n + 1$ and $2n + 3$.

These forms are the gateway to the proofs on integers in section 5.2.

It reads: $2n$ is “two n ” and $2n + 1$ is “two n plus one”; the letter n stands for any integer, which we do not replace with any specific value. In plain words: saying “let n be any integer” announces that everything that follows holds for all integers at once; that is what gives the proofs in section 5.2 their general scope.

Quick questions (unaided)

1. $1 - 6x$: sum or product?
2. $(x + 3)(x - 7)$: sum or product?
3. Give the opposite of $-5x + 4$.
4. Write “three times an integer, plus 1” using the letter n .
5. Write two consecutive even integers.

Answers: 1) sum; 2) product; 3) $5x - 4$; 4) $3n + 1$; 5) $2n$ and $2n + 2$.

Testing a formula is not enough to prove it: the pyramid of cubes

A pyramid has 1 cube on the first layer, 4 cubes on the second, 9 cubes on the third, and so on: layer number k contains k^2 cubes. We denote by S_n the total number of cubes in a pyramid with n layers. “It reads: S_n is ‘ S sub n ’.” In plain words: the subscript specifies the number of layers being considered; thus S_3 is the total for three layers.

1. Calculate S_1, S_2, S_3 and S_4 . We get $S_1 = 1, S_2 = 1 + 4 = 5, S_3 = 1 + 4 + 9 = 14$ and $S_4 = 1 + 4 + 9 + 16 = 30$.
2. Three formulas are suggested:

$$A_n = n^2, \quad B_n = \frac{n(n+1)(2n+1)}{6}, \quad C_n = \left(\frac{n(n+1)}{2}\right)^2.$$

Are the formulas A_n and C_n both disproved as early as $n = 2$? Check that B_n does give the four values from question 1.

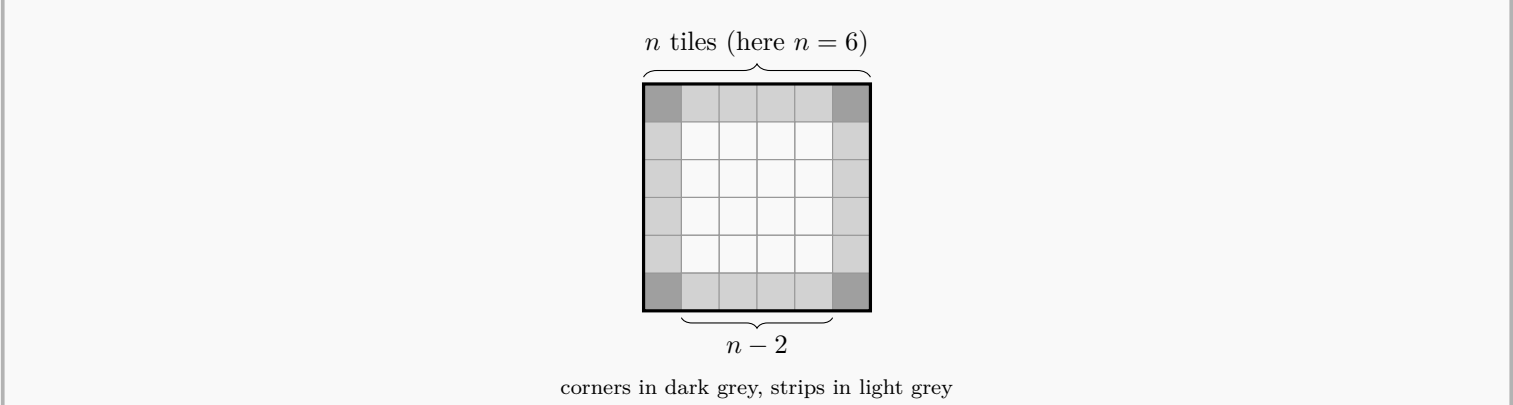
3. Do these four matches prove that $B_n = S_n$ for every integer $n \geq 1$? No: they only show that the formula withstands four tests. A general proof could start from the case $n = 1$, then show that the formula adds exactly n^2 cubes at each step from $n - 1$ to n layers, for every integer $n \geq 2$. “It reads: $n \geq 1$ is ‘ n is greater than or equal to 1’”: n is 1 or more.

4. Assuming the formula $S_n = B_n$, calculate the number of cubes for 20 layers: $S_{20} = \frac{20 \times 21 \times 41}{6} = 10 \times 7 \times 41 = 2870$.

Method takeaway. A single value that fails is enough to rule out a universal formula; values that work, however many, do not prove it. Testing is used to disprove or to check, never to replace a proof.

Guided reading: the four meanings of the “=” sign on the mosaic square

Let $n \geq 2$ be an integer. A square is bordered with mosaic tiles: in the square of size n (n tiles per side), only the outer border is tiled. We denote by C the number of tiles in the border.



One same figure, four uses of the “=” sign:

1. A formula. On each side, between the two corners, $n - 2$ tiles remain; adding the 4 corners gives:

$$C = 4 \times (n - 2) + 4.$$

Here, “=” links two quantities: C depends on n .

2. An identity. By splitting it differently: four strips of $n - 1$ tiles running around the square, each strip taking one corner: we count $C = 4 \times (n - 1)$. Two ways of counting the same tiles, so, for every integer $n \geq 2$:

$$4(n - 2) + 4 = 4(n - 1).$$

Here, “=” means “always equal”. This is a proof by double counting, and algebra confirms it: $4(n - 2) + 4 = 4n - 8 + 4 = 4n - 4 = 4(n - 1)$.

3. An evaluation. For $n = 8$: $C = 4 \times 7 = 28$; for $n = 100$: $C = 4 \times 99 = 396$. Here, “=” gives the resulting value: we substitute for the letter and calculate.

4. An equation. Can such a border be tiled with 482 tiles? We look for all integers n such that $4(n - 1) = 482$: here, “=” expresses a condition on the unknown. Yet $4(n - 1)$ is a multiple of 4 and 482 is not ($482 = 4 \times 120 + 2$): the solution set is empty. With 492 tiles, however, $n - 1 = 123$, so $n = 124$. Depending on the equation and the domain, the solution set may be empty, contain one or more values (possibly infinitely many), or coincide with the entire domain.

Summary. Formula (relationship between quantities), identity (equality stated over an entire domain), evaluation (value obtained by substitution), equation (condition for which all solutions are sought): four uses of the same sign. This chapter focuses mainly on identities; zero-product equations will be studied in Chapter 02.

Facing an “=” sign: what question should you ask?

Status	Question	Example and how to handle it
Formula	Which quantities are linked?	$P = 2\pi R$: for a given circle, P is evaluated from R .
Identity	Why are the two expressions equal for all allowed values?	$2(x + 3) = 2x + 6$: the distributive property transforms the left-hand side into the right-hand side.
Equation	For which value(s) does the equality hold true?	$2x + 3 = 11$: the equality is false for most numbers and true for $x = 4$.
Evaluation	What value is obtained after substitution?	For $x = -2$, $x^2 - 3x = 10$.

To prove an identity, start from one side and transform it through justified equalities until you reach the other. To disprove a claimed identity, a single counterexample (a value for which the equality is false) is enough. To solve an equation, you must find all values that work: a few trials do not guarantee that there are no others.

Quick questions (unaided)

1. $3(x - 2) = 3x - 6$: identity or equation? Justify.

2. $3x - 6 = 0$: identity or equation? Give one value that works and one that does not.

3. Is the statement $(x + 1)^2 = x^2 + 1$ an identity? Disprove it if necessary.

Answers: (1) identity, by the distributive property; (2) equation, true for $x = 2$ and false for instance for $x = 0$; (3) no: for $x = 1$, $(1 + 1)^2 = 4$ whereas $1^2 + 1 = 2$, and $4 \neq 2$, which reads “4 is not equal to 2”.

2 Expanding and simplifying

Expanding

Expanding a product means writing it as an equal sum by carrying out the multiplications. Expanding and simplifying then means writing this sum with as few terms as possible.

In plain words: expanding means removing the brackets from a product by distributing; the expression keeps the exact same value, only the way it is written changes.

2.1 Distributive property (prior learning from Year 9: to review)

Distributive property

For all real numbers k, a, b :

$k(a + b) = ka + kb$ and $k(a - b) = ka - kb$.

Since multiplication is commutative, we can also distribute from the right:

$(a + b)k = ak + bk$ and $(a - b)k = ak - bk$.

“It reads: $k(a + b)$ is ‘ k times a plus b ’, and ka is ‘ k a ’, that is, the product $k \times a$.” In plain words: multiplying a sum by k means multiplying each of its terms by k .

Example

- $-3x(4 - 5x) = -3x \times 4 + (-3x) \times (-5x) = -12x + 15x^2$. Watch the signs: $(-3x) \times (-5x) = +15x^2$.
- $2(4x + 3) - (5 - x) = 8x + 6 - 5 + x = 9x + 1$. The $-$ sign in front of the brackets is a factor of -1 : it gives the opposite of the expression.

Like terms, coefficient, simplifying

Two terms are like terms when they have the exact same literal part: $5x^2$ and $-2x^2$ (literal part x^2), $3x$ and x (literal part x), 4 and -9 (no letter). The number multiplying the literal part is the coefficient of the term: in $-2x^2$, the coefficient is -2 ; in x , it is 1 . Simplifying a sum means collecting its like terms to write it with as few terms as possible; we then order it: the x^2 terms, then the x terms, then the constant term. Exception: after using a notable identity, the result keeps the order of the formula (model D2, part 3).

In plain words: simplifying is neither expanding nor factorising; it means grouping the x^2 terms with the x^2 terms, the x terms with the x terms, and the numbers with the numbers, never mixing them together.

Why do we only add the coefficients?

Proof tool: the distributive property, read from right to left.

$6x - 4x = 6 \times x - 4 \times x = (6 - 4) \times x = 2x$: this is right-distributivity, $ak - bk = (a - b)k$, with $k = x$. The common literal part factors out, and only the coefficients are calculated. Two terms that are not like terms cannot be combined into a single term: for $x = 1$, $3x^2 + 2x$ equals 5 , which might suggest $5x$ or $5x^2$; but for $x = 2$, $3x^2 + 2x$ equals 16 , whereas $5x = 10$ and $5x^2 = 20$. ■

Writing model L3: simplifying an expression

The steps

- ① Identify the like terms: those with the exact same literal part (x^2 , x , or no letter).
- ② Group separately the x^2 terms, the x terms, and then the numbers, writing their coefficients in brackets.
- ③ Add only their coefficients and order the result: x^2 , then x , then the number.

On your paper

Simplify: a) $7x + 4 - 3x - 8$; b) $2x^2 - 5x + x^2 + 8x - 1$.a) ① The like terms are $7x$ and $-3x$, then 4 and -8 .

$$7x + 4 - 3x - 8 = (7 - 3)x + (4 - 8) \quad \text{② I collect like terms}$$

$$= \boxed{4x - 4} \quad \text{③ I add the coefficients}$$

b) ① The like terms are $2x^2$ and x^2 , then $-5x$ and $8x$; the number -1 stands alone.

$$2x^2 - 5x + x^2 + 8x - 1 = (2 + 1)x^2 + (-5 + 8)x - 1 \quad \text{② I collect like terms}$$

$$= \boxed{3x^2 + 3x - 1} \quad \text{③ I add the coefficients}$$

→ Exercise sheet: exercise 1.2 c) (model L3)

2.2 Expanding double brackets (double distributive property)

Expanding double brackets (double distributive property)

For all real numbers a, b, c, d :

$$(a + b)(c + d) = ac + ad + bc + bd.$$

Each term in the first factor multiplies each term in the second: four products.

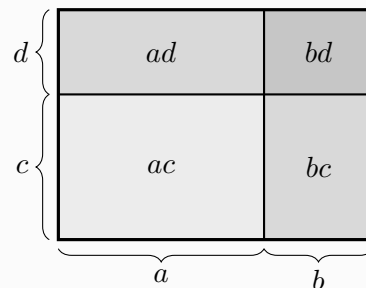
“It reads: $(a + b)(c + d)$ is ‘ a plus b , times c plus d .’ In plain words: we distribute a over c and d , then b over c and d ; the two cross products ad and bc are the ones most often forgotten.”

Geometric view of expanding double brackets (for positive lengths)

Proof tool: double counting of an area (the same surface area calculated in two ways), valid for positive lengths.

Consider a rectangle of length $a + b$ and width $c + d$. Splitting it at distances a and c gives four rectangles (see figure opposite).The area of the large rectangle is $(a + b)(c + d)$. The sum of the areas of the four pieces is $ac + ad + bc + bd$, hence

$$(a + b)(c + d) = ac + ad + bc + bd.$$



This figure explains the four products and makes it impossible to forget the two cross products. Since the dimensions of a rectangle are positive, it only covers that case; the following algebraic proof establishes the identity for all real numbers. ■

Proof of the double distributive property (the $k = a + b$ trick)Proof tool: direct proof by naming a block k , using a previously known property.We want to prove that $(a + b)(c + d) = ac + ad + bc + bd$, using only the single distributive property, which is already known.The idea: treat $(a + b)$ as a single number, which we call k .Step 1. Let $k = a + b$. Then:

$$(a + b)(c + d) = k(c + d) = kc + kd \quad (\text{distributive property, } k \text{ distributed over } c + d).$$

Step 2. Replace k with its value $a + b$, and distribute again:

$$kc = (a + b)c = ac + bc \quad \text{and} \quad kd = (a + b)d = ad + bd \quad (\text{distributive property, twice}).$$

Step 3. Putting the pieces back together:

$$(a + b)(c + d) = ac + bc + ad + bd = ac + ad + bc + bd.$$

In the last equality, we merely swapped the two middle terms: addition is commutative. Expanding double brackets thus follows from three applications of the distributive property, followed by swapping terms in a sum. Therefore, for all real numbers a, b, c and d , $(a + b)(c + d) = ac + ad + bc + bd$. ■

Writing model D1: expanding and simplifying using the distributive property

The steps

- ① Write out all the products: each term in one factor multiplies each term in the other, $k(a + b) = ka + kb$ (two products) and $(a + b)(c + d) = ac + ad + bc + bd$ (four products); write any negative term in brackets.
- ② Calculate each product with its sign; keep the expansion of a product preceded by a $-$ sign inside brackets, then remove those brackets by changing the sign of each of its terms.
- ③ Simplify: collect the x^2 terms, the x terms, the numbers, and order them.
- ④ Check with $x = 1$: the starting expression and the final result must give the same value.

Skip any step that does not apply: in b), there are no like terms to collect, so we jump from ② to ④.

On your paper

Expand and simplify: a) $(2x + 3)(x - 4)$; b) $5(3x - 2)$.

a)

$$(2x + 3)(x - 4) = 2x \times x + 2x \times (-4) + 3 \times x + 3 \times (-4) \quad \textcircled{1}$$

$$= 2x^2 - 8x + 3x - 12$$

② I calculate each product

$$= \boxed{2x^2 - 5x - 12}$$

③ I collect like terms

④ Check ($x = 1$): $5 \times (-3) = -15$ and $2 - 5 - 12 = -15$. ✓

b)

$$5(3x - 2) = 5 \times 3x + 5 \times (-2) \quad \textcircled{1}$$

$$= \boxed{15x - 10}$$

② I calculate each product

④ Check ($x = 1$): $5 \times 1 = 5$ and $15 - 10 = 5$. ✓

Subtracting a product

Subtracting an expression means adding its opposite: $A - B = A + (-B)$. To subtract a product, we therefore write its expansion in brackets, and then remove those brackets by changing the sign of each of its terms (part 1, opposite of an expression):

$$x^2 - (x + 1)(x + 2) = x^2 - (x^2 + 2x + x + 2) = x^2 - x^2 - 2x - x - 2 = -3x - 2.$$

In plain words: the $-$ sign placed in front of a product applies to its entire expansion, and not just to its first term.

Writing model D1 (continued): subtracting a product

Follow the steps of model D1. At step ②, the expansion of a product preceded by a $-$ sign stays inside brackets; the next line removes these brackets by changing the sign of each of its terms. A single $-$ sign in front of brackets is the factor -1 : we write directly $-(4 - x) = -4 + x$.

On your paper

Expand and simplify $2x(x + 5) - (x - 3)(x + 4)$.

$$2x(x + 5) - (x - 3)(x + 4)$$

$$= 2x \times x + 2x \times 5$$

$$- (x \times x + x \times 4 + (-3) \times x + (-3) \times 4) \quad \textcircled{1}$$

$$= 2x^2 + 10x - (x^2 + 4x - 3x - 12)$$

② I calculate each product

$$= 2x^2 + 10x - x^2 - 4x + 3x + 12$$

② the $-$ sign changes the sign of each term in the brackets

$$= \boxed{x^2 + 9x + 12}$$

③ I collect like terms

④ Check ($x = 1$): $2 \times 6 - (-2) \times 5 = 22$ and $1 + 9 + 12 = 22$. ✓

Your turn to complete: $(3x - 2)(4x + 5)$

The critical step is the line with the signs.

$$(3x - 2)(4x + 5) = 3x \times 4x + 3x \times 5 + (\dots) \times 4x + (\dots) \times 5 = 12x^2 + \dots - \dots - 10 = \dots$$

Check for $x = 1$: the product equals $(3 - 2) \times (4 + 5) = \dots$; does your expansion give the same value? This check catches certain mistakes, without replacing the proof, which is the distributive property.

$$\text{Answers: } (-2)(-2) = 4; (-2)(4x) = -8x; (-2)(5) = -10; (3x)(4x) = 12x^2; (3x)(5) = 15x; (4x)(-2) = -8x; (5)(-2) = -10; \text{ for } x = 1, 12 - 8 + 15 - 8 - 10 = 1; \text{ and } 4 + 12 + 15 - 8 - 10 = 9.$$

A line of equalities must remain an equality

Writing $(2x + 3)(x - 4) = 2x^2 - 8x + 3x - 12 = 2x^2 - 5x - 12$ is valid because each step replaces an expression with an expression of equal value. Conversely, never write an instruction, a calculation, and its result in the same chain as though they were equal. Every “=” sign states that both sides have the same value.

▲ Classic pitfalls when expanding

1. Forgetting the cross products: $(x + 2)(x + 5) = x^2 + 10$ is not an identity: for $x = 1$, $(1 + 2) \times (1 + 5) = 18$ whereas $1^2 + 10 = 11$, and $18 \neq 11$. The correct expansion is $x^2 + 7x + 10$.

2. Losing a sign: in $(x - 4)(3x + 2)$, the -4 multiplies both terms in the second pair of brackets. Why this mistake is common: we read “4” and forget that the sign belongs to the term.

3. Half-distributing a $-$ sign: subtracting a bracketed group means adding its opposite, $A - B = A + (-1)B$. Thus $2(4x + 3) - (5 - x) = 8x + 6 - 5 + x$, and not $8x + 6 - 5 - x$: the factor -1 changes the sign of every term in the group.

Quick questions (unaided)

1. Expand and simplify $(x + 4)(x + 2)$.

2. Expand and simplify $(3x - 1)(x + 5)$.

3. Expand and simplify $2(3x - 4) - (5 - x)$.

Answers: 1) $x^2 + 6x + 8$; 2) $3x^2 + 14x - 5$; 3) $6x - 8 - 5 + x = 7x - 13$.

→ Exercise sheet: exercises 1.2 a), 1.3, 1.4, 3.6 a) and 6.1 c) (model D1); exercise 1.5: rework the proof of the double distributive property (writing: model J1, section 5.2)

3 The three notable identities

3.1 Geometric discovery: the square of a sum

We return to the kitchen garden from the introductory activity: a square of side $a + b$ splits naturally into four pieces, as shown in the figure in the proof below.

Geometric proof of $(a + b)^2 = a^2 + 2ab + b^2$ (for $a \geq 0$ and $b \geq 0$)

Proof tool: double counting of an area.

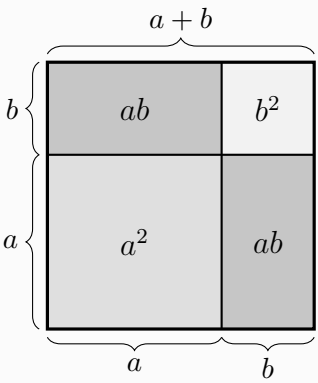
Consider a large square of side $a + b$, divided as in the figure opposite into two squares (of sides a and b) and two congruent rectangles (of dimensions a and b).

1. The area of the large square is $S = (a + b)^2$.

2. The areas of the pieces are: a^2 for the first square, b^2 for the second, and ab for each of the two rectangles.

3. The sum of the four areas is $S' = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$. “It reads: S' is ‘ S prime’”: a second area name, for the sum of the pieces.

4. The four pieces cover the large square exactly, hence $S = S'$. Therefore, for all real numbers $a \geq 0$ and $b \geq 0$, $(a + b)^2 = a^2 + 2ab + b^2$. ■



Honest remark: this proof represents a and b as lengths, so it assumes $a \geq 0$ and $b \geq 0$ (a piece may be degenerate, i.e. flattened with zero area, if a length is zero). The algebraic proof below, on the other hand, holds for all real numbers, including negative ones: that is its strength.

3.2 The three identities, proved

The three notable identities

For all real numbers a and b :

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

The term $2ab$ is called the double product. These equalities are identities: they hold true for all values of a and b .

“It reads: $(a + b)^2$ is ‘ a plus b , all squared’”: the brackets show that the entire sum is squared; “ $a^2 + 2ab + b^2$ reads ‘ a squared, plus two a b , plus b squared’”. In plain words: the square of a sum is not the sum of the squares, you must add the double product; the square of a difference is similar, with the double product subtracted; and the product of the sum and the difference of the same numbers is the difference of their squares. “For all real numbers a and b ” means that a and b can be replaced by any numbers, including negative numbers or decimals, without the equality ceasing to be true.

Algebraic proof of the three identities

Proof tool: direct calculation from the double distributive property.

We simply expand the double brackets: the notable identities are merely expansions learned by heart because they come up constantly.

1. Square of a sum.

$$(a + b)^2 = (a + b)(a + b) = a \times a + a \times b + b \times a + b \times b = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

2. Square of a difference. (Watch the rule of signs: $(-b) \times (-b) = +b^2$.)

$$(a - b)^2 = (a - b)(a - b) = a \times a + a \times (-b) + (-b) \times a + (-b) \times (-b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2.$$

3. Product of a sum and a difference. (The two cross products cancel out.)

$$(a + b)(a - b) = a \times a + a \times (-b) + b \times a + b \times (-b) = a^2 - ab + ab - b^2 = a^2 - b^2.$$

Each calculation uses only the double distributive property, which holds for all numbers. Therefore, for all real numbers a and b , $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$ and $(a + b)(a - b) = a^2 - b^2$. ■

Why keep both proofs?

The geometric proof makes the origin of the double product visible: two rectangles of area ab . However, it assumes that a and b represent positive lengths. The algebraic proof also covers negative values. For example, with $a = -3$ and $b = 5$:

$$(-3 + 5)^2 = 2^2 = 4$$

and

$$(-3)^2 + 2 \times (-3) \times 5 + 5^2 = 9 - 30 + 25 = 4.$$

The diagram gives the intuition; algebra establishes the full domain of validity.

▲ The central trap of the chapter

The equalities $(a + b)^2 = a^2 + b^2$ and $(a - b)^2 = a^2 - b^2$ are not identities.

Numerical counterexample for the first equality (a single one is enough!): for $a = 3$ and $b = 4$:

$$(3 + 4)^2 = 7^2 = 49 \quad \text{whereas} \quad 3^2 + 4^2 = 9 + 16 = 25.$$

What is missing is the double product $2ab = 2 \times 3 \times 4 = 24$, and $25 + 24 = 49$. This counterexample disproves the statement “for all a and b ”; it does not say that the equality fails for every pair of values. For the second equality, finding a counterexample is the subject of exercise 2.4 b) on the exercise sheet. Squaring does not “distribute” over additions: it only distributes over multiplications, since $(ab)^2 = a^2b^2$.

3.3 The square of a difference, seen geometrically: the cut-away square

Geometric proof of $(a - b)^2 = a^2 - 2ab + b^2$ (for $0 < b < a$)

Proof tool: double counting of an area, then isolating the target square.

This time, we start from a large square of side a , and the square under study, of side $a - b$, lies inside it: it is the bottom-left corner in the figure opposite.

1. The large square, of area a^2 , is split into four pieces: the square of area $(a - b)^2$, two rectangles of dimensions b and $a - b$, and a small square of area b^2 . Hence:

$$a^2 = (a - b)^2 + 2b(a - b) + b^2.$$

2. We isolate the cut-away square:

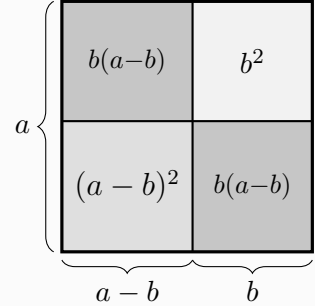
$$(a - b)^2 = a^2 - 2b(a - b) - b^2.$$

3. We expand $2b(a - b) = 2ab - 2b^2$ (distributive property):

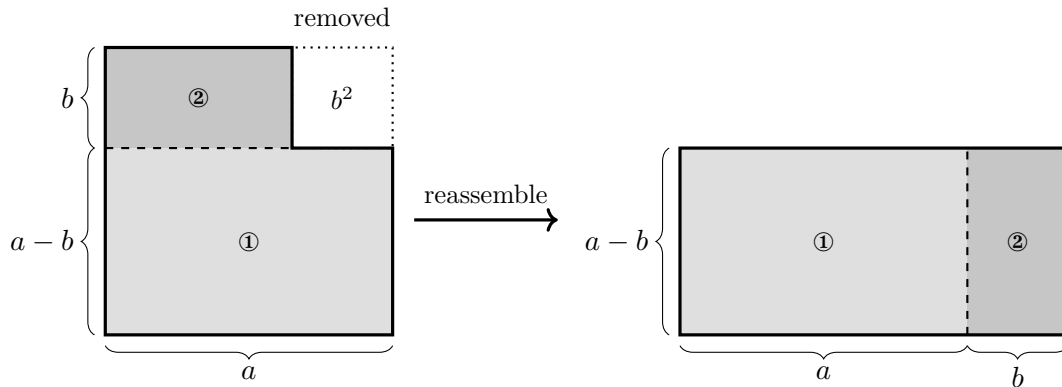
$$(a - b)^2 = a^2 - 2ab + 2b^2 - b^2 = a^2 - 2ab + b^2.$$

Therefore, for all real numbers a and b such that $0 < b < a$, $(a - b)^2 = a^2 - 2ab + b^2$. ■

Visual reading: to obtain the cut-away square, we remove two strips of area ab each ($2b(a - b) + 2b^2 = 2ab$)... but these two strips overlap on the small square b^2 , which is removed twice: so we add it back once. Hence $a^2 - 2ab + b^2$.



3.4 The difference of two squares, seen geometrically: cut and reassemble



piece ①: rectangle $a \times (a - b)$ piece ②: rectangle $b \times (a - b)$, rotated and reattached on the right

Geometric proof of $a^2 - b^2 = (a + b)(a - b)$ (for $0 < b < a$)

Proof tool: cutting and reassembling, preserving the area.

1. On the left: from a square of side a (area a^2), we remove a small square of side b in one corner. The remaining L-shaped region has area $a^2 - b^2$.
2. We cut this L-shape into two pieces along the dashed line: piece ①, a rectangle $a \times (a - b)$, and piece ②, a rectangle $b \times (a - b)$.
3. We rotate piece ② and reattach it to the end of piece ①: we obtain a rectangle of width $a - b$ and length $a + b$, hence of area $(a + b)(a - b)$.
4. Cutting and reassembling does not change the area. Therefore, for all real numbers a and b such that $0 < b < a$, $a^2 - b^2 = (a + b)(a - b)$. ■

3.5 Expanding with a notable identity

Writing model D2: expanding with a notable identity

The steps

- ① Identify a and b : a is the first block, b the second, as whole blocks including coefficients (in $(3x + 5)^2$, $a = 3x$, not x).
- ② Write the formula with these blocks: $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$, $(a + b)(a - b) = a^2 - b^2$; each block is written in brackets, except for a number written with digits alone or a single letter: $(2x)^2$, but 9^2 and x^2 .
- ③ Calculate each part: $(3x)^2 = 9x^2$, the square applies to both the coefficient and the letter.
- ④ Check with $x = 1$: the starting expression and the result must have the same value.
- The identities hold for all numbers a and b : we can therefore replace a and b with entire blocks, such as $2x$. The result keeps the order of the formula, even when the number is the first block. If the expression already contains the letter a , we write the formula with A and B : “I use $(A + B)(A - B) = A^2 - B^2$ with $A = a$ and $B = 3$.”

On your paper

Expand and simplify: a) $(2x + 9)^2$; b) $(5x + 2)(5x - 2)$; c) $(5 - x)^2$.

a) ① I use $(a + b)^2 = a^2 + 2ab + b^2$ with $a = 2x$ and $b = 9$.

$$(2x + 9)^2 = (2x)^2 + 2 \times 2x \times 9 + 9^2 \quad ②$$

$$= \boxed{4x^2 + 36x + 81} \quad ③$$

④ Check ($x = 1$): $11^2 = 121$ and $4 + 36 + 81 = 121$. ✓

b) ① I use $(a + b)(a - b) = a^2 - b^2$ with $a = 5x$ and $b = 2$.

$$(5x + 2)(5x - 2) = (5x)^2 - 2^2 \quad ②$$

$$= \boxed{25x^2 - 4} \quad ③$$

④ Check ($x = 1$): $7 \times 3 = 21$ and $25 - 4 = 21$. ✓

c) ① I use $(a - b)^2 = a^2 - 2ab + b^2$ with $a = 5$ and $b = x$.

$$(5 - x)^2 = 5^2 - 2 \times 5 \times x + x^2 \quad ②$$

$$= \boxed{25 - 10x + x^2} \quad ③$$

④ Check ($x = 1$): $4^2 = 16$ and $25 - 10 + 1 = 16$. ✓

Your turn to complete: $(3x - 8)^2$

I use $(a - b)^2 = a^2 - 2ab + b^2$ with $a = \dots\dots\dots$ and $b = \dots\dots\dots$

$$(3x - 8)^2 = (\dots\dots\dots)^2 - 2 \times \dots\dots\dots \times \dots\dots\dots + (\dots\dots\dots)^2 = \dots\dots\dots\dots\dots\dots$$

Check for $x = 1$: $(3 - 8)^2 = \dots\dots\dots$; does your expanded form give the same value?

Answers: $a = 3x$ and $b = 8$; $(3x)^2 = 9x^2$; $-2 \times 3x \times 8 = -48x$; $8^2 = 64$; for $x = 1$, $(3 - 8)^2 = 25$ and $9 - 48 + 64 = 25$.

Two less standard examples, and a trap

- With fractions: $\left(\frac{1}{3} + x\right)\left(\frac{1}{3} - x\right) = \left(\frac{1}{3}\right)^2 - x^2 = \frac{1}{9} - x^2$.
- With a decimal number: $(y - 3,5)^2 = y^2 - 2 \times y \times 3,5 + 3,5^2 = y^2 - 7y + 12,25$.
- The trap: $(x + 3)(x - 5)$ is neither a square nor a “sum times difference” ($3 \neq 5$): no notable identity applies, so we go back to expanding double brackets, $x^2 - 5x + 3x - 15 = x^2 - 2x - 15$.

Writing model D1 (continued): an expression containing a square

The steps of model D1. When the expression contains a square and other products, we expand each product separately, one line per product (the square using model D2), then substitute back into the expression, placing each expansion in brackets. Only one check, on the whole expression.

On your paper

Expand and simplify $(x + 4)^2 - 3(x - 1)$.

① I expand each product separately.

I use $(a + b)^2 = a^2 + 2ab + b^2$ with $a = x$ and $b = 4$: $(x + 4)^2 = x^2 + 2 \times x \times 4 + 4^2 = x^2 + 8x + 16$.

$3(x - 1) = 3 \times x + 3 \times (-1) = 3x - 3$.

$(x + 4)^2 - 3(x - 1) = (x^2 + 8x + 16) - (3x - 3)$ ② I replace each product with its expansion

$= x^2 + 8x + 16 - 3x + 3$ ② the $-$ sign changes the sign of each term inside the brackets

$= \boxed{x^2 + 5x + 19}$ ③ I collect like terms

④ Check ($x = 1$): $5^2 - 3 \times 0 = 25$ and $1 + 5 + 19 = 25$. ✓

→ Exercise sheet: exercises 2.1, 2.2, 3.5 and 6.1 a) (model D2); 6.3 a) and 6.6 a) (models D2 and D1) 📖 optional: exercises 2.5 to 2.7

3.6 Identity, counterexample: proving or disproving

Identity, equal expressions, counterexample

An identity is an equality between two algebraic expressions that is true for all values of the letters: this is the case for $(a + b)^2 = a^2 + 2ab + b^2$. Two expressions linked by an identity are said to be equal: they have the same value for all values of the letters. A counterexample to a “for all numbers” statement is a specific case where it is false: a value of the letters for which the two sides do not have the same value.

It reads: $9 \neq 5$ “9 is not equal to 5”. In plain words: an identity never breaks down; a single breakdown, a counterexample, is enough for an equality not to be an identity.

Why is a single counterexample enough?

Proof tool: the meaning of the word “all”.

The statement “for all numbers x , $(x + 12)^2 = x^2 + 144$ ” says that the equality is true for every value of x , without exception. For $x = 1$, $(1 + 12)^2 = 169$ whereas $1^2 + 144 = 145$: a single value that does not work contradicts “for all”, so the statement is false. On the other hand, examples that work, even many of them, do not prove an identity: for $x = 0$, $(0 + 12)^2 = 144 = 0^2 + 144$, and yet the equality is not an identity. An identity is proved by a chain of justified equalities (model J1, section 5.2). ■

Writing model T3: disproving a general statement with a counterexample

The steps

① Choose the simplest example: $x = 1$ first (for two letters, $a = 1$ and $b = 1$; for integers, the smallest positive integers that fit), otherwise the simplest one that works.

② Calculate, for this example, what the statement compares: the two sides of an equality, or the number involved.

③ Conclude: the statement is false for this example, so it is not true for all numbers.

Try $x = 1$ first; if both sides are equal for this value, try the next simplest value, and only write down the chosen counterexample on your paper.

On your paper

Show that each statement is false. a) For all numbers x , $(x + 2)^2 = x^2 + 4$. b) For all numbers a and b , $(a + b)^2 = a^2 + ab + b^2$. c) The sum of two consecutive integers is even.

a) ① For $x = 1$, ② $(x + 2)^2 = 3^2 = 9$ whereas $x^2 + 4 = 1 + 4 = 5$: ③ since $9 \neq 5$, the equality is not true for all numbers x .

b) ① For $a = 1$ and $b = 1$, ② $(a + b)^2 = 2^2 = 4$ whereas $a^2 + ab + b^2 = 1 + 1 + 1 = 3$: ③ since $4 \neq 3$, the equality is not true for all numbers a and b .

c) ① For the consecutive integers 1 and 2, ② the sum is $1 + 2 = 3$, and 3 is not even: ③ the statement “the sum of two consecutive integers is even” is not true for all integers.

Writing model T2: answering a “true or false” question

The steps ① State the answer: true or false. ② Justify. Equality between given numbers: calculate both sides separately and compare them. False general statement: give a counterexample. True general statement: quote the course rule or prove it. A statement beginning with “there exists” is proved with a single example.	On your paper True or false? a) For all numbers x, $(3x - 2)^2 = 3x^2 - 2$. b) For all numbers x, $(x + 2)(x - 2) + 4 = x^2$. c) $7^2 + 24^2 = 25^2$. d) There exists a number x for which $(x + 1)^2 = x^2 + 1$. a) ① <u>FALSE</u>. ② For $x = 2$, $(3x - 2)^2 = 4^2 = 16$ whereas $3x^2 - 2 = 12 - 2 = 10$: since $16 \neq 10$, the equality is not true for all numbers x. b) ① <u>TRUE</u>. ② I expand the left-hand side. I use $(a + b)(a - b) = a^2 - b^2$ with $a = x$ and $b = 2$: $(x + 2)(x - 2) + 4 = x^2 - 2^2 + 4 = x^2 - 4 + 4 = x^2$. Therefore, for all numbers x, $(x + 2)(x - 2) + 4 = x^2$. ■ c) ① <u>TRUE</u>. ② I calculate both sides separately: $7^2 + 24^2 = 49 + 576 = 625$ and $25^2 = 625$. Both sides are equal. d) ① <u>TRUE</u>. ② One example is enough: for $x = 0$, $(0 + 1)^2 = 1$ and $0^2 + 1 = 1$: both sides are equal for $x = 0$, so such a number exists.
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→ Exercise sheet: exercise 2.4 (model T2; a), b), e) and f): model T3; d): model J1); exercises 3.6 b), 4.5 a), b) and d), 5.4 f) and 6.5 c) (model T3)

Quick questions (unaided)

1. Expand $(x + 6)^2$.

2. Expand $(5x - 2)^2$.

3. Expand $(6x + 5)(6x - 5)$.

4. True or false: $(a - b)^2 = a^2 - 2ab - b^2$? Give a numerical counterexample if it is false.

Answers: 1) $x^2 + 12x + 36$; 2) $25x^2 - 20x + 4$; 3) $36x^2 - 25$; 4) False: for $a = 1$ and $b = 1$, $(a - b)^2 = 0$ whereas $a^2 - 2ab - b^2 = 1 - 2 - 1 = -2$.

4 Factorising

Factorising

To factorise an expression means to rewrite it as a product equal to the original expression. An expression is in factorised form when its main operation is a multiplication. We then use the distributive property or a notable identity in the opposite direction to expanding. A factorisation is generally not unique.

In plain words: factorising means going backwards from expanding and returning to a product; the expression keeps the exact same value, only the way it is written changes.

4.1 First tool: the common factor

Factoring out a common factor

For all real numbers k , a , b :

$$ka + kb = k(a + b) \quad \text{and} \quad ka - kb = k(a - b).$$

We say that we have factored out k . The common factor k can be a number, a letter or a whole expression in brackets.

It reads from left to right: $ka + kb = k(a + b)$ “ k a plus k b equals k times a plus b in brackets”. In plain words: what is common to both terms comes out in front of the brackets; what remains of each term is written inside.

Why does factoring out not change the value?

Proof tool: the distributive property, read from right to left.

The distributive property gives $k(a + b) = ka + kb$ for all numbers k , a and b . An equality can be read both ways: $ka + kb = k(a + b)$. Factorising therefore means applying the property used for expanding, from right to left; that is why we check a factorisation by expanding it again. For example, $6x + 10 = 2 \times 3x + 2 \times 5 = 2(3x + 5)$, and expanding again gives $2(3x + 5) = 6x + 10$. ■

Valid factorisation or complete factorisation?

$12x^2 - 18x = 3x(4x - 6)$ and $12x^2 - 18x = 6x(2x - 3)$ are both valid factorisations. The second extracts the largest integer factor common to the coefficients (their HCF, defined in chapter 00) and the full common power of x : it is more complete. Unless instructed otherwise, we look for this form, then check by expanding again.

Writing model F1: factorising using a common factor (number, letter)

The steps

- ① Find the factor common to all terms: the highest common factor of the coefficients, and each letter present in all terms, with its smallest exponent.
- ② Write each term as a product showing this factor, and underline the common factor.
- ③ Write the common factor in front of brackets and, inside the brackets, what remains of each term, with its sign: $ka + kb = k(a + b)$ and $ka - kb = k(a - b)$.
- ④ Check by expanding the resulting product.

When the problem specifies the common factor, step ① states it: “The problem specifies the common factor $4x$: $4x$ divides $16x^2$ and $28x$.”

On your paper

Factorise: a) $14x^2 - 21x$; b) $6 + 3x$; c) $x^2 + 9x$; d) $12a + 20b$.

a) ① The common factor is $7x$: 7 is the highest common factor of 14 and 21, and x is present in both terms, with the smallest exponent 1.

$$14x^2 - 21x = \underline{7x} \times 2x - \underline{7x} \times 3 \quad \text{② I bring out the common factor}$$

$$= \boxed{7x(2x - 3)} \quad \text{③ I factor out } 7x$$

④ Check: $7x(2x - 3) = 14x^2 - 21x$. ✓

b) ① The common factor is 3: 3 is the highest common factor of 6 and 3, and x is not present in the term 6.

$$6 + 3x = \underline{3} \times 2 + \underline{3} \times x \quad \text{② I bring out the common factor}$$

$$= \boxed{3(2 + x)} \quad \text{③ I factor out } 3$$

④ Check: $3(2 + x) = 6 + 3x$. ✓

c) ① The common factor is x : 1 and 9 have no common divisor other than 1, and x is present in both terms, with the smallest exponent 1.

$$x^2 + 9x = \underline{x} \times x + \underline{x} \times 9 \quad \text{② I bring out the common factor}$$

$$= \boxed{x(x + 9)} \quad \text{③ I factor out } x$$

④ Check: $x(x + 9) = x^2 + 9x$. ✓

d) ① The common factor is 4: 4 is the highest common factor of 12 and 20, and no letter is present in both terms.

$$12a + 20b = \underline{4} \times 3a + \underline{4} \times 5b \quad \text{② I bring out the common factor}$$

$$= \boxed{4(3a + 5b)} \quad \text{③ I factor out } 4$$

④ Check: $4(3a + 5b) = 12a + 20b$. ✓

→ Exercise sheet: exercise 3.1 (model F1)

Why can we factor out a bracketed expression?

Proof tool: a bracketed expression seen as a single number, like the block $k = a + b$ in section 2.2.

A bracketed expression like $(x - 5)$ represents a single number, whatever the value of x . The property $ka + kb = k(a + b)$ therefore applies with $k = (x - 5)$:

$$(x - 5)(x + 1) + (x - 5)(x + 4) = (x - 5)[(x + 1) + (x + 4)] = (x - 5)(2x + 5).$$

It reads: $[(x + 1) + (x + 4)]$ “open square bracket, $x + 1$ plus $x + 4$, close square bracket”. In plain words: square brackets act as brackets that contain other brackets. A square also contains the common factor, since $(x - 5)^2 = (x - 5) \times (x - 5)$. ■

Writing model F2: factorising when the common factor is a bracketed expression

The steps

- ① Spot the bracketed expression common to both terms; a square A^2 means $A \times A$ and contains the factor A .
- ② Write the common bracketed expression in front of square brackets and, inside the square brackets, what remains of each term, in round brackets, with the sign preceding it.
- ③ Simplify the inside of the square brackets: a $-$ sign in front of brackets changes the sign of each of its terms; if a number is common to all resulting terms, factor it out.
- ④ Check with $x = 1$: the starting expression and the factorised form must have the same value.

On your paper

Factorise, then simplify: a) $(2x+1)(x+4) - (2x+1)(3x-2)$; b) $(x-6)^2 + (x-6)(2x+1)$; c) $(3x+2)^2 - 4(3x+2)$.

a) ① The bracketed expression common to both terms is $(2x+1)$.

$$(2x+1)(x+4) - (2x+1)(3x-2)$$

$$= (2x+1)[(x+4) - (3x-2)] \quad \text{② I factor out } (2x+1)$$

$$= (2x+1)(x+4-3x+2)$$

③ the $-$ sign changes the sign of each term inside the brackets

$$= (2x+1)(-2x+6)$$

③ I simplify inside the brackets

$$= \boxed{2(2x+1)(-x+3)}$$

③ I factor out 2

④ Check ($x = 1$): $3 \times 5 - 3 \times 1 = 12$ and $2 \times 3 \times 2 = 12$. ✓

b) ① The bracketed expression common to both terms is $(x-6)$, since $(x-6)^2 = (x-6) \times (x-6)$.

$$(x-6)^2 + (x-6)(2x+1)$$

$$= (x-6)[(x-6) + (2x+1)] \quad \text{② I factor out } (x-6)$$

$$= (x-6)(x-6+2x+1) \quad \text{③ I remove the inner brackets}$$

$$= \boxed{(x-6)(3x-5)} \quad \text{③ I simplify inside the brackets}$$

④ Check ($x = 1$): $(-5)^2 + (-5) \times 3 = 10$ and $(-5) \times (-2) = 10$. ✓

c) ① The bracketed expression common to both terms is $(3x+2)$, since $(3x+2)^2 = (3x+2) \times (3x+2)$.

$$(3x+2)^2 - 4(3x+2) = (3x+2)[(3x+2) - 4] \quad \text{② I factor out } (3x+2)$$

$$= \boxed{(3x+2)(3x-2)} \quad \text{③ I simplify inside the brackets}$$

④ Check ($x = 1$): $5^2 - 4 \times 5 = 5$ and $5 \times 1 = 5$. ✓

Your turn to complete: $(x+3)(5x-1) - (x+3)(2x+4)$

The bracketed expression common to both terms is We write:

$$(x+3)(5x-1) - (x+3)(2x+4) = (x+3)[(5x-1) - (\dots\dots\dots)] = (x+3)(\dots\dots\dots) = (x+3)(\dots\dots\dots).$$

The critical step: the $-$ sign applies to the entire second bracketed expression.

$$\text{Answers: } (x+3)(5x-1) - (x+3)(2x+4) = (x+3)(5x-1-2x-4) = (x+3)(3x-5) = (x+3)(3x-5) \quad \text{check for } x=1: 4 \times 4 - 4 \times 6 = -8 \text{ and } 4 \times (-2) = -8.$$

→ Exercise sheet: exercise 3.2 (model F2)

4.2 Second tool: recognising an identity

Perfect square, difference of two squares

An expression is a perfect square when it is equal to the square of another expression: $x^2 + 24x + 144 = (x+12)^2$. An integer is a perfect square when it is the square of an integer, such as $49 = 7^2$. A difference of two squares is an expression of the form $a^2 - b^2$, such as $x^2 - 16 = x^2 - 4^2$. Squares to know: 1; 4; 9; 16; 25; 36; 49; 64; 81; 100; 121; 144; and $9x^2 = (3x)^2$, $25t^2 = (5t)^2$...

It reads: $a^2 - b^2$ “ a squared minus b squared”. In plain words: a difference of two squares has two terms, two squares separated by a $-$ sign; the square of a sum or of a difference, when expanded, has three terms, two squares at the ends and the double product in the middle.

Why does an identity allow us to factorise, and why expand again?

Proof tool: an equality can be read both ways.

The three identities from section 3.2 are true for all numbers; read from right to left, they turn a sum into a product:

$$a^2 - b^2 = (a + b)(a - b), \quad a^2 + 2ab + b^2 = (a + b)^2, \quad a^2 - 2ab + b^2 = (a - b)^2.$$

To use them, we must recognise a and b : hence the step that takes a “snapshot” of the structure (number of terms, squares, double product). If the middle term is not equal to $2ab$ exactly, the expression is not a perfect square. We check by expanding again: this algebraic equality establishes that the factorisation gives back the starting expression for every value of the letter. Substituting a number can catch a mistake, but agreement for a single value does not prove an identity. ■

Writing model F3: factorising with a notable identity

The steps

① Take a snapshot of the structure: two terms that are squares, separated by a $-$ sign, is a difference of two squares; three terms whose two outer terms are squares may be a perfect square.

② Write each square in the form $(\dots)^2$ to identify a and b ; for three terms, check that the middle term is equal to $2 \times a \times b$ exactly and note its sign.

③ Apply the identity, keeping blocks in brackets: $a^2 - b^2 = (a + b)(a - b)$, $a^2 + 2ab + b^2 = (a + b)^2$, $a^2 - 2ab + b^2 = (a - b)^2$.

④ Simplify the inside of each factor; if a number is common to all terms of a factor, factor it out.

⑤ Check by expanding the resulting product.

When the order is reversed, a is the first term written: in $1 - 10x + 25x^2$, $1 = 1^2$ and $25x^2 = (5x)^2$, so $1 - 10x + 25x^2 = (1 - 5x)^2$.

On your paper

Factorise: a) $9x^2 - 100$; b) $16x^2 - 56x + 49$; c) $4x^2 - 64$.

a) ① Two terms separated by a $-$ sign, and both are squares: it is a difference of two squares. ② $9x^2 = (3x)^2$ and $100 = 10^2$, so $a = 3x$ and $b = 10$.

$$\begin{aligned} 9x^2 - 100 &= (3x)^2 - 10^2 & \text{②} \\ &= \boxed{(3x + 10)(3x - 10)} & \text{③ } a^2 - b^2 = (a + b)(a - b) \end{aligned}$$

⑤ Check: $(3x + 10)(3x - 10) = (3x)^2 - 10^2 = 9x^2 - 100$. ✓

b) ① Three terms, and the two outer terms are squares: it may be a perfect square. ② $16x^2 = (4x)^2$ and $49 = 7^2$, so $a = 4x$ and $b = 7$; the double product is $2 \times 4x \times 7 = 56x$: this is indeed the middle term, preceded by a $-$ sign.

$$\begin{aligned} 16x^2 - 56x + 49 &= (4x)^2 - 2 \times 4x \times 7 + 7^2 & \text{②} \\ &= \boxed{(4x - 7)^2} & \text{③ } a^2 - 2ab + b^2 = (a - b)^2 \end{aligned}$$

⑤ Check: $(4x - 7)^2 = (4x)^2 - 2 \times 4x \times 7 + 7^2 = 16x^2 - 56x + 49$. ✓

c) ① Two terms separated by a $-$ sign, and both are squares: it is a difference of two squares. ② $4x^2 = (2x)^2$ and $64 = 8^2$, so $a = 2x$ and $b = 8$.

$$\begin{aligned} 4x^2 - 64 &= (2x)^2 - 8^2 & \text{②} \\ &= (2x + 8)(2x - 8) & \text{③ } a^2 - b^2 = (a + b)(a - b) \\ &= 2(x + 4) \times 2(x - 4) = \boxed{4(x + 4)(x - 4)} & \text{④ I factor out 2 from each factor} \end{aligned}$$

⑤ Check: $4(x + 4)(x - 4) = 4(x^2 - 4^2) = 4x^2 - 64$. ✓

Your turn to complete: factorise $81x^2 - 4$

Two terms separated by a $-$ sign, and both are squares: $81x^2 = (\dots)^2$ and $4 = (\dots)^2$. Therefore:

$$81x^2 - 4 = (\dots + \dots)(\dots - \dots).$$

Check by expanding again:

$$\text{Answers: } 81x^2 = (9x)^2 \text{ and } 4 = 2^2; 81x^2 - 4 = (9x)^2 - 2^2 = (9x + 2)(9x - 2); \text{ check: } (9x + 2)(9x - 2) = 81x^2 - 4.$$

Writing model F3 (continued): a squared block, a missing number

The steps of model F3.

When a or b is a block in brackets, we keep it whole inside square brackets; the check expands both forms down to the same simplified expression.

When the problem already states that the expression is a perfect square and asks for a missing number, we follow steps ②, ③ and ⑤.

On your paper

a) Factorise $(2x + 5)^2 - 49$. b) Find a missing number so that $x^2 + \dots x + 121$ is a perfect square.

a) ① Two terms separated by a $-$ sign, and both are squares: it is a difference of two squares.

② $(2x + 5)^2$ is the square of the block $2x + 5$ and $49 = 7^2$, so $a = 2x + 5$ and $b = 7$.

$$(2x + 5)^2 - 49 = (2x + 5)^2 - 7^2 \quad \text{②}$$

$$= [(2x + 5) + 7][(2x + 5) - 7] \quad \text{③ } a^2 - b^2 = (a + b)(a - b)$$

$$= (2x + 12)(2x - 2) \quad \text{④ I simplify each factor}$$

$$= 2(x + 6) \times 2(x - 1) = \boxed{4(x + 6)(x - 1)} \quad \text{④ I factor out 2 from each factor}$$

⑤ Check: $4(x + 6)(x - 1) = 4(x^2 - x + 6x - 6) = 4x^2 + 20x - 24$ and $(2x + 5)^2 - 49 = 4x^2 + 20x + 25 - 49 = 4x^2 + 20x - 24$. ✓

b) $x^2 + \dots x + 121$: ② $x^2 = x^2$ and $121 = 11^2$, so $a = x$ and $b = 11$; the double product is $2 \times x \times 11 = 22x$. ③ The missing number can be $\boxed{22 \text{ or } -22}$: $x^2 + 22x + 121 = (x + 11)^2$ and $x^2 - 22x + 121 = (x - 11)^2$.

⑤ Check: $(x + 11)^2 = x^2 + 2 \times x \times 11 + 11^2 = x^2 + 22x + 121$ and $(x - 11)^2 = x^2 - 2 \times x \times 11 + 11^2 = x^2 - 22x + 121$. ✓

▲ When no identity applies

Can we factorise $x^2 - 10$? The structure is indeed “square minus something”... but 10 is not the square of any integer ($3^2 = 9$ and $4^2 = 16$), and no number known in this chapter has 10 as its square (we take this as given). So we cannot yet name b : this roadblock is temporary. In chapter 02, the positive number whose square is 10 will be defined and denoted by $\sqrt{10}$, which reads “square root of 10”. In plain words: $\sqrt{10} \times \sqrt{10} = 10$, and $\sqrt{10}$ lies between 3 and 4. We will then be able to write

$$x^2 - 10 = x^2 - (\sqrt{10})^2 = (x + \sqrt{10})(x - \sqrt{10}).$$

On the other hand, $x^2 + 64$ (with a $+$) is not a difference of two squares, nor a perfect square, which would have three terms: none of the three identities turns a sum of two squares into a product, and this roadblock is not temporary within the scope of this chapter. A mathematical verdict always specifies the tools and framework within which it is stated.

Quick questions (unaided)

1. Factorise $9x - 54$.

2. Factorise $x^2 - 100$.

3. Factorise $x^2 + 16x + 64$.

4. Factorise $9x^2 - 64$.

5. Can $x^2 + 9$ be factorised using a notable identity?

Answers: 1) $9(x - 6)$; 2) $(x + 10)(x - 10)$; 3) $(x + 8)^2$; 4) $(3x + 8)(3x - 8)$; 5) no: $x^2 + 9$ is a sum of two squares; it is neither a difference of two squares nor a perfect square, which would have three terms.

→ Exercise sheet: exercises 2.3, 3.3, 3.4, 3.5, 3.6 c), 6.1 b), 6.6 b) and 9.1 b) (model F3); exercise 3.6 (models D1, T3 and F3) 🧑 optional: exercise 3.7 (grouping terms, method given in the problem)

One expression, three forms, three pieces of information

Consider

$$J = (5x + 1)^2 - 16.$$

1. Starting form. It shows a difference between a square and 16. For $x = -\frac{1}{5}$, the block $5x + 1$ equals 0, so $J = -16$ without expanding.
2. Expanded form: $J = 25x^2 + 10x + 1 - 16 = 25x^2 + 10x - 15$. It collects the expression into three ordered terms.
3. Factorised form.

$$J = (5x + 1)^2 - 4^2 = [(5x + 1) + 4][(5x + 1) - 4] = (5x + 5)(5x - 3) = 5(x + 1)(5x - 3).$$

It immediately shows that $J = 0$ for $x = -1$: one factor is then zero. Zero-product equations will be covered in chapter 02; here, we simply observe the information made visible.

4. Cross-check for $x = -\frac{1}{5}$:

$$25 \times \frac{1}{25} + 10 \times \left(-\frac{1}{5}\right) - 15 = 1 - 2 - 15 = -16 \quad \text{and} \quad 5 \times \left(-\frac{1}{5} + 1\right) \times (-1 - 3) = 5 \times \frac{4}{5} \times (-4) = -16.$$

No single form is “best” in absolute terms: the right form is the one that reveals the desired information with the fewest operations.

Writing model V1 (continued): choosing the right form

The steps of model V1 (part 1). When the expression is given in several forms, a sentence chooses the form before step ①, and explains why.

On your paper

We know that $B = 2x^2 - x - 28 = (x - 4)(2x + 7)$. Calculate B for $x = 4$, choosing the quickest form.

I choose the factorised form, because for $x = 4$ the factor $x - 4$ equals 0: the product is zero without any further calculation. ① $B = (x - 4)(2x + 7)$. ② For $x = 4$:

$$B = (4 - 4) \times (2 \times 4 + 7) \quad \text{②}$$

$$= 0 \times 15 = 0 \quad \text{③}$$

④ For $x = 4$, $\boxed{B = 0}$.

→ Exercise sheet: exercises 6.3 c) and 6.6 c) and d) (model V1, with choice of form)

5 Clever mental maths, and proving with letters

5.1 Identities as a speed tool

Clever calculation

Calculating cleverly means rewriting a calculation without changing its value, so that the remaining operations are simpler. A “round” number (10, 100, 1 000...) is often a clue: a number close to a round number can be written as “round number plus or minus a small number”, such as $105 = 100 + 5$. However, the clue alone does not determine the tool: notable identity or distributive property.

In plain words: you do not calculate faster by calculating harder, but by choosing a form where only easy calculations remain.

Why are identities useful for mental maths?

Proof tool: an identity is true for all numbers.

The identities from section 3.2 are true for all numbers a and b , and therefore also for $a = 100$ and $b = 5$:

$$105^2 = (100 + 5)^2 = 100^2 + 2 \times 100 \times 5 + 5^2 = 10\,000 + 1\,000 + 25 = 11\,025.$$

With a round a and a small b , each term can be worked out in your head: three seconds of calculation, without a calculator. The result is exact, because the identity is proved for all numbers: it is not an approximation. ■

Writing model C1: mentally calculating a square or a product close to a round number

The steps

① Write each number as a round number a plus or minus a small number b , then name the tool: $(a+b)^2$, $(a-b)^2$, $(a+b)(a-b)$, or the distributive property when only one factor is close to a round number.

② Apply the formula with these numbers.

③ Work out each term in your head, then find the result.

④ Check: the units digit and the order of magnitude.

For $(a-b)^2$, same layout: $298^2 = (300-2)^2 = 90\,000 - 1\,200 + 4 = 88\,804$. With decimal numbers, the check looks at the last digit: $20,2 \times 19,8 = 20^2 - 0,2^2 = 399,96$ ends like $2 \times 8 = 16$, in 6.

On your paper

Calculate mentally: a) 302^2 ; b) 302×298 ; c) 23×999 .

a) ① I use $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 300$ and $b = 2$, since $302 = 300 + 2$.

$$302^2 = (300+2)^2 = 300^2 + 2 \times 300 \times 2 + 2^2 \quad ②$$

$$= 90\,000 + 1\,200 + 4 = \boxed{91\,204} \quad ③$$

④ Check: the units digit is that of $2 \times 2 = 4$, and 91 204 is slightly larger than $300^2 = 90\,000$. ✓

b) ① I use $(a+b)(a-b) = a^2 - b^2$ with $a = 300$ and $b = 2$, since $302 = 300 + 2$ and $298 = 300 - 2$.

$$302 \times 298 = (300+2)(300-2) = 300^2 - 2^2 \quad ②$$

$$= 90\,000 - 4 = \boxed{89\,996} \quad ③$$

④ Check: the units digit is that of $2 \times 8 = 16$, namely 6, and 89 996 is slightly smaller than $300^2 = 90\,000$. ✓

c) ① I use the distributive property $k(a-b) = ka - kb$ with $k = 23$, $a = 1\,000$ and $b = 1$, since $999 = 1\,000 - 1$ and only 999 is close to a round number.

$$23 \times 999 = 23 \times (1\,000 - 1) = 23 \times 1\,000 - 23 \times 1 \quad ②$$

$$= 23\,000 - 23 = \boxed{22\,977} \quad ③$$

④ Check: the units digit is that of $3 \times 9 = 27$, namely 7, and 22 977 is slightly smaller than $23 \times 1\,000 = 23\,000$. ✓

Applicable does not always mean advantageous

Any product of two numbers u and v can, over the real numbers, be centred around their mean:

$$uv = \left(\frac{u+v}{2} - \frac{u-v}{2}\right) \left(\frac{u+v}{2} + \frac{u-v}{2}\right) = \left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2.$$

Thus, $47 \times 999 = 523^2 - 476^2$ is a true equality, but it makes the calculation harder. The distributive property $47 \times 999 = 47 \times (1\,000 - 1) = 46\,953$ is more economical here. A method is “clever” only if it is both exact and less costly.

Writing model C2: calculating mentally by recognising the expanded form of an identity

The steps

① Recognise the form and identify a and b : $a^2 - b^2$, $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$.

② Write the corresponding product: $(a+b)(a-b)$, $(a+b)^2$ or $(a-b)^2$.

③ Work out the brackets in your head, then the single remaining multiplication.

④ Check the units digit.

A difference of two squares becomes a single multiplication; a perfect square becomes the square of a round number. With a $-$ sign, same layout: $63^2 - 2 \times 63 \times 3 + 3^2 = (63-3)^2 = 60^2 = 3\,600$. With decimal numbers, $9,5^2 - 0,5^2 = (9,5+0,5)(9,5-0,5) = 10 \times 9 = 90$; the check looks at the decimal places: $9,5^2$ and $0,5^2$ both end in 25 after the decimal point, as $5 \times 5 = 25$, so their difference has no digits after the decimal point.

On your paper

Calculate mentally: a) $76^2 - 24^2$; b) $38^2 + 2 \times 38 \times 2 + 2^2$.

a) ① I use $a^2 - b^2 = (a+b)(a-b)$ with $a = 76$ and $b = 24$.

$$76^2 - 24^2 = (76+24)(76-24) \quad ②$$

$$= 100 \times 52 = \boxed{5\,200} \quad ③$$

④ Check: 76^2 ends like $6 \times 6 = 36$, in 6; 24^2 ends like $4 \times 4 = 16$, in 6; the difference therefore ends in 0, like 5 200. ✓

b) ① I use $a^2 + 2ab + b^2 = (a+b)^2$ with $a = 38$ and $b = 2$.

$$38^2 + 2 \times 38 \times 2 + 2^2 = (38+2)^2 \quad ②$$

$$= 40^2 = \boxed{1\,600} \quad ③$$

④ Check: the units digits of the three terms are 4, 2 and 4, and $4+2+4 = 10$ ends in 0, like 1 600. ✓

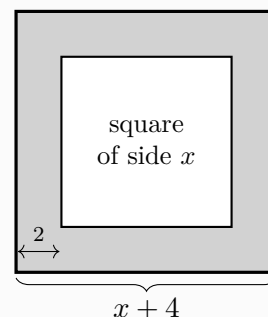
→ Exercise sheet: exercises 4.1, 4.4 a) to c), 4.5 c) and 6.4 a) and b) (model C1); exercises 4.2, 4.3 a), 4.4 d) and 6.4 c) (model C2)

A square border: expanding or factorising an area

An inner square has side length x metres. A border 2 m wide surrounds it: the large square therefore has side length $x + 4$.

The area of the border is denoted by $\mathcal{A}$, which reads “script A”. In plain words: $\mathcal{A}$ is the name given to this area, in square metres. It is the difference of the areas:

$$\mathcal{A} = (x + 4)^2 - x^2.$$



Two transformations give two readings: by expanding, and then, more directly, by factorising,

$$\mathcal{A} = x^2 + 8x + 16 - x^2 = 8x + 16 \quad \text{and} \quad \mathcal{A} = [(x + 4) + x][(x + 4) - x] = (2x + 4) \times 4 = 8(x + 2).$$

For $x = 6$, the geometric check gives $10^2 - 6^2 = 100 - 36 = 64$, and the factorised form gives $8 \times (6 + 2) = 64$.

Writing model T1: writing out the solution to a problem

The steps

- ① Write down what is known: the given data, with their units, converted into compatible units; draw a diagram if needed.
- ② Write down what you are looking for, and the tool: the operation, justified by a sentence or a formula.
- ③ Calculate, using the appropriate model.
- ④ Conclude: a sentence answering the question, with the unit.
- ⑤ Check: order of magnitude, plausibility or reverse calculation.

On your paper

A square photograph of side 15 cm is glued in the centre of a square piece of cardboard of side 17 cm. Calculate mentally the area of the visible part of the cardboard.

- ① I know that the cardboard is a square of side 17 cm and that the photograph, glued in the centre, is a square of side 15 cm.
- ② I am looking for the area $\mathcal{A}$ of the visible part of the cardboard: it is the area of the cardboard minus the area of the photograph, so $\mathcal{A} = 17^2 - 15^2$, in cm^2 .
- ③ I use $a^2 - b^2 = (a + b)(a - b)$ with $a = 17$ and $b = 15$: $\mathcal{A} = (17 + 15)(17 - 15) = 32 \times 2 = 64$.
- ④ Conclusion: the visible part of the cardboard has an area of $\boxed{64 \text{ cm}^2}$.
- ⑤ Check: the visible part is a strip 1 cm wide around the photograph; its centre line measures about $4 \times 16 = 64$ cm, giving an area of about $64 \times 1 = 64 \text{ cm}^2$. ✓

→ Exercise sheet: exercise 4.3 a) and b) (model T1; calculation of a): model C2) 🧩 optional: exercise 4.6

5.2 Proving a general result: from conjecture to proof

The prover's toolbox, algebra version

This chapter contains several complete proofs. Each proof box indicates on its first line the tool it uses; here are the four main tools.

1. Direct proof using algebra. We choose an arbitrary number, denoted by a letter that is never replaced, and we chain together justified equalities (distributive property, identity, simplifying) up to the form that proves the claim: $5(n + 2)$ for a multiple of 5, $2k + 1$ for an odd number. This is the tool used throughout this section.
2. Double counting of an area. We calculate the same area in two ways; the two results are equal. This tool makes the origin of a formula visible, but it only holds for positive lengths: the algebraic proof completes the work (Part 3).
3. Naming a block. We give a name to a whole expression ($k = a + b$) so as to apply an already known result to it, then we expand the block again (Section 2.2).
4. Counterexample. To disprove an equality claimed “for all numbers”, a single numerical example is enough: $(3 + 4)^2 = 49$ whereas $3^2 + 4^2 = 25$. Conversely, a finite list of favourable examples does not prove the statement for all real numbers.

Note: testing a simple value ($x = 0$ or $x = 1$) can spot a mistake, but does not prove an identity for every x ; an identity is proved by a chain of equalities valid for all values.

Conjecturing is not proving

A conjecture is a statement put forward on the basis of observations and which remains to be proved. For a statement about all the numbers in an infinite set, a finite list of examples establishes only the tested cases: it does not cover the universal statement. In the situations in this section, the proof chooses an arbitrary number, specifies what the letter stands for, and then chains together justified equalities. On the other hand, a single counterexample is enough to disprove a universal statement.

It reads: “for every integer n ”: the statement is true whichever integer is chosen. In plain words: a conjecture is an idea to be proved; until the proof is written down, it remains an idea, even after a thousand successful trials.

Writing model J3: testing a statement on examples and conjecturing

The steps

- ① Calculate each example, writing out the calculation.
- ② Write down what you observe; if the problem asks for it, state the conjecture: “It seems that...”.
- ③ State what the examples establish: only the tested cases; for all numbers, a proof is needed.

On your paper

Calculate the sum of two consecutive integers on three examples, state a conjecture, and then say what these examples establish.

- ① $5 + 6 = 11$; $12 + 13 = 25$; $40 + 41 = 81$.
- ② These three sums are odd. It seems that the sum of two consecutive integers is always odd.
- ③ These examples establish only the tested cases: the statement is about infinitely many integers, so a proof using a letter is required.

→ Exercise sheet: exercises 5.1 d), 5.2 a), 5.4 a) and c) and 6.5 a) (model J3)

Calculation program

A calculation program is a sequence of instructions to be carried out in order, starting from an input number. With an arbitrary starting number, denoted by x , the program produces an algebraic expression; the result obtained for a given number is the value of this expression for that number.

In plain words: an algebraic expression is a calculation program written on a single line; $2(3x + 1)$ reads “choose x , multiply it by 3, add 1, multiply the result by 2”.

Why use brackets when translating a program?

Proof tool: order of operations.

The result of an instruction is a single number. When the next instruction multiplies it or squares it, that entire number is multiplied: we put it in brackets. With the program “choose x , add 6, multiply by 4”, the result is $4(x + 6)$; without brackets, $4 \times x + 6$ would only multiply x by 4. For $x = 1$, the program gives $(1 + 6) \times 4 = 28$, whereas $4 \times 1 + 6 = 10$. ■

Writing model P1: using a calculation program

The steps

- ① Start from the starting number: the given number, or x for an arbitrary number.
- ② Carry out the instructions in order, one calculation per instruction; with x , put the previous result in brackets when the next instruction multiplies it or squares it.
- ③ Write down the result; with x , expand and simplify the expression obtained.

With a number, the calculations are written on one line, separated by “;”; with x , use one line per instruction.

On your paper

Program: choose a number; multiply it by -3 ; add 8; multiply the result by 2.
a) Calculate the result starting with 4. b) Check that the starting number 2 gives 4. c) Express the result in terms of the starting number x .

a) ① Starting with 4: ② $4 \times (-3) = -12$; $-12 + 8 = -4$; $-4 \times 2 = -8$. ③ The result is $\boxed{-8}$.

b) ① Starting with 2: ② $2 \times (-3) = -6$; $-6 + 8 = 2$; $2 \times 2 = 4$. ③ I do get 4. ✓

c) ① Starting with x :

② I multiply by -3 : $-3x$.

I add 8: $-3x + 8$.

I multiply the result by 2: $2(-3x + 8)$.

③ The result is $2(-3x + 8) = 2 \times (-3x) + 2 \times 8 = \boxed{-6x + 16}$.

Why can we perform the same operation on both sides?

Proof tool: two equal numbers are one and the same number.

If $A = B$, both sides denote the same number; the same calculation applied to this number gives the same result: $A + c = B + c$, $A - c = B - c$, $A \times c = B \times c$, and, if $c \neq 0$, $\frac{A}{c} = \frac{B}{c}$. In this way we transform the equality “expression = desired result”, one operation per line, until x is isolated. We never divide by 0: that is why we specify $c \neq 0$. The final check, which applies the program to the number found, verifies that no calculation error has been made. ■

Writing model P2: finding the starting number that gives a desired result

The steps

- ① Write the equality “program expression = desired result”.
- ② Solve: perform the same operation on both sides, one per line, until you get $x = \dots$
- ③ Conclude with a sentence answering the question.
- ④ Check by applying the program to the number found.

On your paper

With the same program, whose result is $-6x + 16$: what starting number must be chosen to obtain 22?

- ① I look for x such that $-6x + 16 = 22$.

$$-6x + 16 = 22$$

$$-6x = 22 - 16 = 6 \quad \text{② I subtract 16 from both sides}$$

$$x = \frac{6}{-6} = -1 \quad \text{② I divide both sides by } -6$$

- ③ Conclusion: we must choose $\boxed{-1}$ as the starting number.

- ④ Check: $(-1) \times (-3) = 3$; $3 + 8 = 11$; $11 \times 2 = 22$. ✓

→ Exercise sheet: exercises 5.1 a) to c), 5.2 and 5.3 a) to c) (model P1); exercises 5.1 c) and 5.3 d) (model P2)

Two programs: trials suggest, algebra proves

Program A: choose a number; multiply it by 5; add 7; subtract the starting number; add 3.

Program B: choose a number; multiply it by 2; add 5; multiply the result by 2.

With 1, 2 and then 3, both programs give 14, 18 and 22 respectively. This suggests that they are equivalent, but only deals with three numbers.

Let x be an arbitrary number. Program A gives $5x + 7 - x + 3 = 4x + 10$; Program B gives $2(2x + 5) = 4x + 10$. Both expressions are equal to $4x + 10$. Therefore, for every number x , the two programs give the same result. ■ Here, the proof is not the list of trials; it is the simplification of both results to the same expression.

Why does a chain of equalities prove an identity?

Proof tool: direct proof using algebra.

Each equality in the chain applies a rule that is true for all numbers (distributive property, notable identity, simplifying): each link is therefore true for every value of x , and the chain connects the left-hand side to the right-hand side for every value of x . This is the question to ask when facing an identity (Part 1, table “Facing an = sign”): we start from one side and transform it through justified equalities until we obtain the other, or we transform each side into the same expression. Numerical trials, on the other hand, only cover the values tried. ■

Writing model J1: proving that an equality is true for every number

The steps

- ① Start from one side, usually the one that can be expanded or simplified.
- ② Transform it through a chain of justified equalities, one step per line, until you get the other side (or transform each side into the same expression).
- ③ Conclude: “Therefore, for every number $x, \dots$ ”, then ■.

In a proof, a product is expanded in a single line, with the individual products already worked out.

On your paper

Prove that, for every number x : a) $(x+2)(x+6) - (x+4)^2 = -4$; b) $(3x+1)^2 - 1 = 3x(3x+2)$.

- a) ① I expand the left-hand side.

$$(x+2)(x+6) - (x+4)^2$$

$$= x^2 + 6x + 2x + 12 - (x^2 + 8x + 16) \quad \text{② expanding double brackets, then } (a+b)^2 = a^2 + 2ab + b^2$$

$$= x^2 + 6x + 2x + 12 - x^2 - 8x - 16 \quad \text{② the } - \text{ sign changes the sign of each term in the brackets}$$

$$= -4 \quad \text{② I collect like terms}$$

- ③ I obtain the right-hand side. Therefore, for every number x ,

$$(x+2)(x+6) - (x+4)^2 = -4. \quad \blacksquare$$

- b) ① I expand each expression.

$$(3x+1)^2 - 1 = 9x^2 + 6x + 1 - 1 = 9x^2 + 6x \quad \text{② } (a+b)^2 = a^2 + 2ab + b^2$$

$$3x(3x+2) = 9x^2 + 6x \quad \text{② distributive property}$$

- ③ Both expressions are equal to $9x^2 + 6x$. Therefore, for every number x , $(3x+1)^2 - 1 = 3x(3x+2)$. ■

→ Exercise sheet: exercises 1.5, 2.4 d), 4.3 c), 5.1 d), 5.2 b) and 6.3 b) (model J1)

Complete example: the difference of the squares of two consecutive even integers

Question. What can be said about the difference of the squares of two consecutive even integers?

Step 1: experiment, using a spreadsheet or by hand. For an integer n , the two consecutive even integers $2n$ and $2n + 2$ give the difference $(2n + 2)^2 - (2n)^2$:

n	0	1	4	12	125
$2n$	0	2	8	24	250
$2n + 2$	2	4	10	26	252
$(2n + 2)^2 - (2n)^2$	4	12	36	100	1 004

(In a spreadsheet: A2 = n , B2 = $(2 \times A2 + 2)^2 - (2 \times A2)^2$, then drag down.)

Step 2: conjecture. All five results are multiples of 4: $4 = 4 \times 1$, $12 = 4 \times 3$, $36 = 4 \times 9$, $100 = 4 \times 25$ and $1\,004 = 4 \times 251$. It seems that the difference is always a multiple of 4. The calculated cases are established, but the statement covering all integers has not yet been proved.

Step 3: prove. Let n be an arbitrary integer; the two consecutive even integers are $2n$ and $2n + 2$. Two methods:

Method 1 (factorising via $a^2 - b^2$):

$$(2n + 2)^2 - (2n)^2 = [(2n + 2) + 2n][(2n + 2) - 2n] = (4n + 2) \times 2 = 8n + 4 = 4(2n + 1).$$

Method 2 (expanding):

$$(2n + 2)^2 - (2n)^2 = 4n^2 + 8n + 4 - 4n^2 = 8n + 4 = 4(2n + 1).$$

Now n is an integer, so $2n + 1$ is an integer: $4(2n + 1)$ is a multiple of 4.

Step 4: conclude and check. Therefore, for every integer n , the difference of the squares of two consecutive even integers is a multiple of 4; better still, it equals $4(2n + 1)$, four times an odd number. ■ Check against the table: for $n = 12$, $4(2n + 1) = 4 \times 25 = 100$. ✓

Comparison of the two methods: expanding is more automatic; factorising also reveals the reason behind the result: the difference is the product of the sum of the two numbers, $4n + 2$, and the gap between them, 2. Factorising is the approach that generalises best (on the exercise sheet: what if the integers are consecutive odd numbers?).

Writing model J2: proving a property of integers

The steps

- ① Choose the letter and state what it stands for: “Let n be an arbitrary integer”, then write the numbers in the problem using it (consecutive n , $n + 1$; even $2n$; odd $2n + 1$).
- ② Calculate, expand or factorise: a chain of equalities, one step per line, without ever replacing n with a number.
- ③ Recognise the form obtained: “Now...” ($3k$ is a multiple of 3, $2k + 1$ is odd), stating why k is an integer.
- ④ Conclude: “Therefore, for every integer n , ...”, then ■.

On your paper

Prove that the sum of four consecutive even integers is a multiple of 4.

① Let n be an arbitrary integer. Four consecutive even integers are written $2n$, $2n + 2$, $2n + 4$ and $2n + 6$.

② I calculate their sum:

$$\begin{aligned} 2n + (2n + 2) + (2n + 4) + (2n + 6) &= 8n + 12 & \text{② I collect like terms} \\ &= 4(2n + 3) & \text{② I factor out 4} \end{aligned}$$

③ Now n is an integer, so $2n + 3$ is an integer: $4(2n + 3)$ is a multiple of 4.

④ Therefore, for every integer n , the sum of four consecutive even integers is a multiple of 4. ■

→ Exercise sheet: exercises 5.4 d) and e) and 6.5 b) (model J2)

Quick questions (unaided)

1. Calculate 49×51 mentally.
2. Calculate $59^2 - 41^2$ using a single multiplication.
3. Write two consecutive integers the larger of which is n .
4. “ $n^2 + n + 41$ is prime for $n = 1, 2$ and 3 ”: does this prove that it is prime for every integer n ?

Answers: 1) $(50 - 1)(50 + 1) = 2\,500 - 1 = 2\,499$; 2) $(59 + 41)(59 - 41) = 100 \times 18 = 1\,800$; 3) $n - 1$ and n ; 4) no, the three calculations establish only the three tested cases (and for $n = 41$, $41^2 + 41 + 41 = 41 \times 43$ is not prime).

→ Exercise sheet: cross-topic block “Justifying and reasoning” optional: exercises 5.5 to 5.8

6 Common mistakes in this chapter

▲ Common mistakes

1. $(a + b)^2 = a^2 + b^2$: the classic mistake. Why it is made: thinking that “squaring” distributes like multiplication. Counterexample: $(3 + 4)^2 = 49 \neq 25$. The double product $2ab$ is not optional.

2. $(2x)^2 = 2x^2$: the square applies to the whole factor: $(2x)^2 = 4x^2$. Why: seeing only the letter, not the coefficient.

3. Writing $-(3 - 8x) = -3 - 8x$: the $-$ sign must change every term: $-(3 - 8x) = -3 + 8x$. Why: stopping at the first term.

4. Factorising $x^2 - 169$ as $(x - 13)^2$: confusing $a^2 - b^2$ (two terms) with $(a - b)^2$ (which, when expanded, has three terms). Check by expanding again: $(x - 13)^2 = x^2 - 26x + 169$, not $x^2 - 169$.

5. Thinking that $5x + 4$ is a product “because there is $5x$ ”: the main operation is $+$, so it is a sum and nothing can be simplified in it.

6. Stopping at $3x(3x - 2)$ when factorising $9x^2 - 6x + 1$: forgetting the $+1$, the expression was not $9x^2 - 6x$; in fact, $9x^2 - 6x + 1 = (3x - 1)^2$.

7. Thinking that checking three examples proves a property for all integers: those trials establish three cases only; here, the proof must reason on an arbitrary integer and justify each step.

→ Exercise sheet: brevet-style problems 6.1 and 6.3 to 6.6; exercise 9.1 (models V1 and F3), to be revisited in Chapter 02

→ Exercise sheet: 📖 optional: exercises 6.2, 6.7 to 6.9, 7.1 to 7.4, 8.1 to 8.6 and 9.2

Summary sheet

Chapter summary: recognise / calculate / conclude		
I recognise...	... so I calculate...	... and I check
a product $k(a + b)$ or $(a + b)(c + d)$	I expand (distributive property or expanding double brackets), then I simplify	test value $x = 1$
$(a + b)^2$	$a^2 + 2ab + b^2$	the double product is there; test value $x = 1$
$(a - b)^2$	$a^2 - 2ab + b^2$	$-$ sign only on $2ab$; test value $x = 1$
$(a + b)(a - b)$	$a^2 - b^2$	two terms only; test value $x = 1$
a sum $ka + kb$	I factorise using the common factor k (number, letter or bracket)	expand again mentally; common bracket: test value $x = 1$
two squares separated by $-$	$a^2 - b^2 = (a + b)(a - b)$	expand again mentally
three terms, outer squares, middle $= 2ab$	$(a + b)^2$ if $+2ab$, $(a - b)^2$ if $-2ab$	check the double product, then expand again mentally
201^2 , 197×203 or $84^2 - 16^2$...	nearby round number (C1) or recognised expanded form (C2)	units digit (and order of magnitude for a square or a product)
“prove that for every integer...”	algebra: $2n$ even, $2n + 1$ odd, $3k$ multiple of 3, n , $n + 1$ consecutive; then expand or factorise	conclusion starting with “Therefore, for every integer n ...”; optional numerical check after the proof

Quick-fire questions (without help)

1. Expand $(x + 15)^2$.

2. Factorise $x^2 - 400$.

3. Factorise $9x^2 - 36$.

4. Calculate 301^2 mentally.

5. A multiple of 3 is written...? An odd number is written...?

Answers. 1) $x^2 + 30x + 225$; 2) $(x + 20)(x - 20)$; 3) $(3x + 6)(3x - 6)$; 4) $90\,000 + 600 + 1 = 90\,601$; 5) $3k$; $2n + 1$.

Self-assessment (multiple-choice) – only one correct answer per question

1. The expression $3(x - 7)$ is:
a. a sum b. a product c. a square d. an equation

2. $(x + 7)^2$ is equal to:
a. $x^2 + 49$ b. $x^2 + 7x + 49$ c. $x^2 + 14x + 49$ d. $2x + 14$

3. $(3x - 2)^2$ is equal to:
a. $9x^2 - 4$ b. $9x^2 - 12x + 4$ c. $3x^2 - 12x + 4$ d. $9x^2 + 12x + 4$

4. $(x + 15)(x - 15)$ is equal to:
a. $x^2 - 225$ b. $x^2 + 225$ c. $x^2 - 30x - 225$ d. $x^2 - 30x + 225$

5. The factorised form of $x^2 - 196$ is:
a. $(x - 14)^2$ b. $(x + 14)(x - 14)$ c. $x(x - 196)$ d. impossible

6. The factorised form of $x^2 - 24x + 144$ is:
a. $(x - 12)^2$ b. $(x + 12)^2$ c. $(x + 12)(x - 12)$ d. $(x - 24)(x + 12)$
7. The opposite of $6 - 5x$ is:
a. $6 + 5x$ b. $-6 - 5x$ c. $5x - 6$ d. $\frac{1}{6 - 5x}$

8. To calculate 203×197 , which notable identity directly transforms $(200 + 3)(200 - 3)$ into a difference of squares?
a. $(a + b)^2 = a^2 + 2ab + b^2$ b. $(a - b)^2 = a^2 - 2ab + b^2$
c. $(a + b)(a - b) = a^2 - b^2$ d. expanding double brackets

9. The common factor of $(x - 4)(x + 1) + (x - 4)(2x + 3)$ is:
a. x b. $(x + 1)$ c. $(x - 4)$ d. $(2x + 3)$

10. The sum of two consecutive integers n and $n + 1$ equals:
a. $2n$ b. $2n + 1$ c. $n^2 + 1$ d. $2n + 2$

Answers: 1) b 2) c 3) d 4) a 5) b 6) c 7) a 8) d 9) c 10) b

Progressive reference glossary

Map of writing models

Code	Type of exercise	The steps, in brief	Exercise sheet
L1	Sum or product?	last operation, nature, terms or factors	1.1, block c)
L2	Opposite of an expression	$-(\dots)$, change the sign of each term, zero sum	1.2 b), 6.1 c)
L3	Simplify	like terms, grouping, added coefficients	1.2 c)
D1	Expand using the distributive property	all products, each with its sign, simplifying, check at $x = 1$	1.2 a), 1.3, 1.4, 3.6 a), 6.1 c), 6.3 a), 6.6 a), block a)
D2	Expand using an identity	a and b , formula with blocks, each part, check at $x = 1$	2.1, 2.2, 3.5, 6.1 a), 6.3 a), 6.6 a), block a)
F1	Common factor	common factor, make it appear, bring it to the front, expand again	3.1
F2	Common bracket	common bracket, square brackets, simplify inside the square brackets, check at $x = 1$	3.2, block a)
F3	Factorise using an identity	structure, a and b , identity, simplify the factors, expand again	2.3, 3.3, 3.4, 3.5, 3.6 c), 6.1 b), 6.6 b), 9.1 b)
C1	Square or product near a round number	round number and small gap, formula, mental calculation, units and order of magnitude	4.1, 4.4 a) to c), 4.5 c), 6.4 a) b)
C2	Recognised expanded form	form, product, mental calculation, units digit	4.2, 4.3 a), 4.4 d), 6.4 c)
V1	Value of an expression	copy out, substitute (negative in brackets), order of operations, sentence	6.3 c), 6.6 c) d), 9.1 a) c)
P1	Calculation program	starting number, one instruction per calculation, result	5.1 a) to c), 5.2, 5.3 a) to c)
P2	Starting number	equality, same operation on both sides, conclusion, check	5.1 c), 5.3 d)
J1	Equality for every number	start from one side, justified chain, “Therefore, for every”	1.5, 2.4 d), 4.3 c), 5.1 d), 5.2 b), 6.3 b)
J2	Property of integers	“Let”, calculation, “Now”, “Therefore, for every integer”	5.4 d) e), 6.5 b), block d)
J3	Examples and conjecture	calculations, observation and conjecture, scope of the examples	5.1 d), 5.2 a), 5.4 a) c), 6.5 a)
T1	Problem	what is known, what is sought, calculation, conclusion, check	4.3 a) b)
T2	True or false	stated answer, justification depending on the case	2.4, block c)
T3	Counterexample	simplest example, both sides, conclusion	2.4 a) b) e) f), 3.6 b), 4.5 a) b) d), 5.4 f), 6.5 c), block b)

“block”: the cross-topic block “Justifying and reasoning”, after exercise 5.8.

Reading symbols: how to say them aloud

Algebraic notation is not a secret code: it shortens a sentence about numbers. This table is a directory to consult as needed, not a list to memorise before starting. First read the problem in plain English; then come back to the symbol to shorten it. For each notation, the last column also explains what it is used for in this chapter.

Symbol	How it reads	What it means and what it is used for
$3x \quad ab$	“three x ”, “ a b ”	the $\times$ sign is understood between a number and a letter, between two letters or in front of a bracket: $3x = 3 \times x$, $ab = a \times b$, $2ab = 2 \times a \times b$. Use: reading a double product $2ab$ as a single term.
x^2	“ x squared”	$x \times x$; similarly $E^2 = E \times E$ for a whole expression E . Use: the outer squares of an identity, such as $9x^2 = (3x)^2$.
$(a + b)^2$	“ a plus b , all squared”	the brackets apply to the whole sum: it is $(a + b) \times (a + b)$, not $a^2 + b^2$. Use: recognising a square of a sum before expanding.
$k(a + b)$	“ k times the bracket a plus b ”	the product of the number k and the sum $a + b$; the bracket forms a block. Use: the distributive property $k(a + b) = ka + kb$, in both directions.
$-(a - b)$	“the opposite of the difference of a and b ”	the $-$ sign in front of the bracket is a factor of -1 : $-(a - b) = -a + b$. Use: subtracting a whole expression without losing a sign.
$[\quad]$	“square brackets”	square brackets play the role of brackets that contain other brackets: $(x - 5)[(x + 1) + (x + 4)]$. Use: factoring out a bracket (model F2), factorising a squared block (model F3).
$\frac{x}{2}$	“ x over two”	half of x ; we have $\left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$. Use: identities with fractions.

Symbol	How it reads	What it means and what it is used for
$=$	“equals”	four uses: formula, identity, evaluation, equation (Part 1). Use: asking yourself on every line “why do these two sides have the same value?”.
$\neq$	“is not equal to”	the two sides do not have the same value: $(3 + 4)^2 \neq 3^2 + 4^2$. Use: writing a counterexample.
$\geq \quad \leq \quad <$	“greater than or equal to”, “less than or equal to”, “strictly less than”	$d^2 \geq 0$: a square is always non-negative (non-negative: ≥ 0 ; positive: > 0); $0 < b < a$: b lies strictly between 0 and a . Use: specifying the allowed values, such as $n \geq 2$ for the mosaic square or $0 < b < a$ for an area proof.
$2n \quad 2n + 1$	“two n ”, “two n plus one”	with n an integer, an even number and an odd number; $3k$ is a multiple of 3; $n, n + 1, n + 2$ are three consecutive integers. Use: translating a statement into algebra before proving it.
S_n	“ S subscript n ”	a quantity that depends on the integer n ; S_3 is its value for $n = 3$. Use: naming a total that changes with n .
$1 + 3 + \cdots + (2n - 1)$	“one plus three, and so on up to two n minus one”	the three dots replace the intermediate terms. Use: writing a sum whose number of terms depends on n (exercise 8.2 on the exercise sheet, optional).
$\sqrt{10}$	“square root of ten”	the positive number whose square is 10; it will be defined in Chapter 02. Use: factorising $x^2 - 10$, later on.
$\mathcal{A}$	“script A”	an area. Use: the area of a border, $\mathcal{A} = (x + 4)^2 - x^2$, or of the visible part of a piece of cardboard (model T1).
$\xrightarrow{\text{expand}}$	“gives, by expanding”	moving from one form to another form with the same value. Use: the product / sum diagram in Part 1.
■	“end of proof”	indicates that the proof is complete and the conclusion is established. Use: closing a written proof neatly.